THE FINITENESS OF I WHEN $R[X]/I$ IS FLAT

BY

JACK OHM([1]) AND DAVID E. RUSH

ABSTRACT. Let R be a commutative ring with identity, let X be an indeterminate, and let I be an ideal of the polynomial ring $R[X]$. Let min I denote the set of elements of I of minimal degree and assume henceforth that min I contains a regular element. Then $R[X]/I$ is a flat R-module implies I is a finitely generated ideal. Under the additional hypothesis that R is quasi-local integrally closed, the stronger conclusion that I is principal holds. (An example shows that the first statement is no longer valid when min I does not contain a regular element.)

Let $c(I)$ denote the content ideal of I, i.e. $c(I)$ is the ideal of R generated by the coefficients of the elements of I. A corollary to the above theorem asserts that $R[X]/I$ is a flat R-module if and only if I is an invertible ideal of $R[X]$ and $c(I) = R$. Moreover, if R is quasi-local integrally closed, then the following are equivalent: (i) $R[X]/I$ is a flat R-module; (ii) $R[X]/I$ is a torsion free R-module and $c(I) = R$; (iii) I is principal and $c(I) = R$.

Let ξ denote the equivalence class of X in $R[X]/I$, and let $\langle 1, \xi, \cdots, \xi^t \rangle$ denote the R-module generated by $1, \xi, \cdots, \xi^t$. The following statements are also equivalent: (i) $\langle 1, \xi, \cdots, \xi^t \rangle$ is flat for all $t \geq 0$; (ii) $\langle 1, \xi, \cdots, \xi^t \rangle$ is flat for some $t \geq 0$ for which $1, \xi, \cdots, \xi^t$ are linearly dependent over R; (iii) $I = (f_1, \cdots, f_n)$, $f_i \in$ min I, and $c(I) = R$; (iv) $c(\min I) = R$. Moreover, if R is integrally closed, these are equivalent to $R[X]/I$ being a flat R-module. A certain symmetry enters in when ξ is regular in $R[\xi]$, and in this case (i)–(iv) are also equivalent to the assertion that $R[\xi]$ and $R[1/\xi]$ are flat R-modules.

The main results of this paper are found in §§2 and 3. Many of the difficulties of these sections already occur when the ring R is an integral domain, and the reader might benefit by first confining his attention to this case.

Additional technical difficulties arise when one proceeds to the case that R is an arbitrary ring and I is subject to the restriction that min I contain a regular element. §4 is devoted to a discussion of what happens when one removes this condition on min I. In particular, we make there a conjecture as to the class of rings R with the property that for every ideal I of $R[X]$, if $R[X]/I$ is R-flat, then I is finitely generated. The corresponding question for finitely generated modules is easily answered as follows: The class of rings R with the property that $R[X]/I$ *is a finite flat R-module implies I is finitely generated* is exactly

the class of rings for which finite flat R-modules are projective (and hence includes domains, quasi-local rings, and noetherian rings).

The final section of the paper discusses briefly the case of a polynomial ring in more than one variable and, in particular, Nagata's work on this question.

0. **Notation and terminology.** Ring will always mean commutative ring with identity, and domain will mean a ring without zero-divisors. A non-zero-divisor of a ring will be called a regular element. Ring homomorphisms will always be assumed to map the identity into the identity.

R denotes a ring, X a single indeterminate, and I an ideal of the polynomial ring $R[X]$. When the ring under discussion is a domain, we sometimes emphasize this fact by using D rather than R. R, m is used to signify that R is quasi-local with maximal ideal m. $T(R)$ denotes the total quotient ring of R, and $\overline{R}$ denotes the integral closure of R (in $T(R)$). When $R = \overline{R}$, we say that R is integrally closed. If $R \subset R'$ are rings and P is a prime ideal of R, then $(R')_P$ denotes the ring $(R')_S$, where $S = R \setminus P$. (Here $\setminus$ denotes set-complement.) We use $J(R)$ for the intersection of the maximal ideals of R.

If R and R' are rings and $\phi: R \rightarrow R'$ is a ring homomorphism, then R' may be considered an R-algebra with defining homomorphism ϕ. The following notational conventions allow us to avoid explicit mention of ϕ and to treat R' as if it were an overring of R. For any subsets A of R and A' of R', we define $AA' = \{\Sigma\, \phi(a_i)a_i' \mid a_i \in A,\ a_i' \in A'\}$; similarly, $A' \cap R$ is defined to be $\phi^{-1}(A')$. $R' = R[\xi]$ signifies that the element ξ of R' generates R' as an R-algebra. Whenever we regard $R'[X]$ as an $R[X]$-algebra, we shall have in mind the defining homomorphism ϕ_X obtained by applying ϕ to the coefficients of elements of $R[X]$. Then for an ideal I of $R[X]$, $IR'[X] =$ the ideal of $R'[X]$ generated by $\phi_X(I)$. Note also that $R'[X]$ is isomorphic as an R'-algebra to $R' \otimes_R R[X]$ via the map $r' \otimes f(X) \rightarrow r' \cdot f(X)$.

By "module" we shall always mean a unitary module. An R-module M is called torsion-free if whenever $rm = 0$, $r \in R$, $m \neq 0 \in M$, then r is a zero-divisor of R. We sometimes say that M is R-torsion-free (or R-flat, R-projective, R-free, etc.) to emphasize that we are regarding M as an R-module. If $m_1, \cdots, m_t$ are generators of M, we write $M = \langle m_1, \cdots, m_t \rangle$. When M is R-free, $\mathrm{rk}_R M$ denotes the rank of M. If S is a multiplicative system (m.s.) of R, then M_S denotes the R_S-module $R_S \otimes_R M$; when M is an ideal of R, we shall identify M_S with MR_S.

Let H be a nonempty subset of $R[X]$. $c_R(H)$ (or merely $c(H)$ if the R is clear) denotes the ideal of R generated by the coefficients of the elements of H. More generally, if U is a subring of R, $c_U(H)$ denotes the U-submodule of R generated by the coefficients of the elements of H. Let $f \neq 0 \in R[X]$. If $f = a_0 + a_1 X + \cdots + a_n X^n$ with $a_n \neq 0$, then a_n is called the leading coefficient of f, and

we write $\deg_X f = n$. When there is no possibility of confusion, we shall omit the subscript X. Similarly, if $a_0 = a_1 = \cdots = a_{i-1} = 0$ and $a_i \neq 0$ we write $\text{subd } f = i$. More generally, if H is a nonempty, nonzero subset of $R[X]$, we define $\deg H = \min\{\deg f \mid f \neq 0 \in H\}$ and $\text{subd } H = \min\{\text{subd } f \mid f \neq 0 \in H\}$. For any ideal I of $R[X]$, we define $\min I$ to be $\{f \neq 0 \in I \mid \deg f \leq \deg g \text{ for all } g \neq 0 \in I\}$. Note that $\min I \cup \{0\}$ is then an R-module and that, for any $f_1, f_2 \in \min I$, $a_1 f_2 = a_2 f_1$, where $a_i = $ leading coefficient f_i.

We shall call an ideal A of R *locally trivial* if for every prime ideal P of R, either $AR_P = 0$ or $AR_P = R_P$. Similarly, an ideal I of $R[X]$ will be called *locally principal* (resp., *locally finitely generated*) at primes of $R[X]$ if for every prime ideal P' of $R[X]$, $IR[X]_{P'}$ is principal (resp., finitely generated).

Our primary reference for terminology will be Zariski-Samuel [ZS]. In particular, $\subset$ denotes containment and $<$ strict containment. Also, $(A : B)_C$ denotes $\{c \in C \mid cB \subset A\}$, whenever this makes sense; the subscript C will be dropped when it is clear from the context.

1. Preliminaries on content and flatness.

1.1 Content. Let L be a free R-module with basis $B = \{b_i\}_{i \in I}$, and let U be a subring of R. If $l = \sum a_i b_i \in L$, $a_i \in R$, the U-content of l with respect to B, denoted $c_U(l)$, is defined to be the U-module $\sum a_i U$. $c_U(l)$ is a finitely generated U-module; and when $U = R$, $c_U(l)$ is independent of the choice of basis B. The content has the following properties:

(a) For any $l \in L$ and any ideal A of R, $l \in AL \Leftrightarrow c_U(l) \subset A$.

(b) $c_U(rl) = r \cdot c_U(l)$ for $r \in R$, $l \in L$.

If H is any nonempty subset of L, we define $c_U(H)$ to be the U-module contained in R and generated by $\{c_U(h) \mid h \in H\}$.

If C is a polynomial ring over R (in any number of indeterminates), then C is, in particular, a free R-module with basis consisting of power-products of the indeterminates, and hence C has a U-content function defined with respect to this basis. This function has the following multiplicative property:

(c) *Content formula* [**No**]. For any $f, g \in C$, $c_U(fg)c_U(g)^n = c_U(f)c_U(g)^{n+1}$ for all sufficiently large n.

With one exception, in the proof of 2.17, we only deal with the case that $R = U$; so we shall confine our further discussion to this case. It follows from the content formula that an element $g \in C$ is a zero-divisor of C if and only if there exists $r \neq 0 \in R$ such that $r \cdot c_R(g) = 0$. Another immediate consequence of the content formula is that $c_R(gh) = c_R(h)$ if $c_R(g) = R$.

We also need some properties of the content formula under localization. If R' is an R-algebra and L is a free R-module with basis $\{b_i\}_{i \in I}$, then $L' = R' \otimes_R L$ is a free R'-module with basis $\{1 \otimes b_i\}_{i \in I}$ [**L**, p. 418, Proposition 8].

For any $l \in L$, $l = \Sigma r_i b_i$, $r_i \in R$; and hence $1 \otimes l = \Sigma r_i(1 \otimes b_i)$. Therefore, $c_R(l)R' = c_{R'}(1 \otimes l)$. More generally, if K is a submodule of L and K' is the submodule of L' generated by $\{1 \otimes l \mid l \in K\}$, then $c_R(K) \cdot R' = c_{R'}(K')$. In particular, if S is a multiplicative system of R, then $c_R(K)R_S = c_{R_S}(K_S)$ $(= c_R(K)_S$ by notation).

We also need, in the proofs of 1.5, 1.6, and 2.17, the following observation. Let G be a multiplicative system of the polynomial ring C with the property that $g \in G \Rightarrow c_R(g) = R$, and suppose in C_G, $f/g_1 = (h/g_2)(l/g_3)$, f, h, $l \in C$, $g_i \in G$. Then $c_R(f) = c_R(hl)$; and if $c_R(h) = R$, then $c_R(f) = c_R(l)$. For, there exist g, $g' \in G$ such that $gf = g'hl$, and then the asssertion follows from the content formula. (The awkwardness here is due to the fact that C_G is not necessarily a free R-module and hence does not have a content function in the sense of our definitions. One can get around such problems by working in the more general setting of a not necessarily free R-algebra which has a content function satisfying (a), (b), (c) above. See [OR] for a detailed study of such matters.)

1.2 **Flatness criteria.** Let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules with L free. Then the following are equivalent:

(a) M is flat.

(b) For every ideal A of R and every $r \in R$, $(A : r)M \supset AM : r$ (where $AM : r = \{m \in M \mid rm \in AM\}$).

(c) For every finitely generated ideal A of R, $K \cap AL \subset AK$.

(d) For every $x \in K$, $x \in c(x)K$.

(a) $\Leftrightarrow$ (c) follows from [B(a), p. 33, Corollary],

(a) $\Leftrightarrow$ (b) from [B(a), p. 65, Exercise 22]; and

(a) $\Leftrightarrow$ (d) is easy [B(a), p. 65, Exercise 23]. Our basic reference for flatness will be [B(a)].

We shall also use the following observations: Let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules, and let R' be an R-algebra. If M is R-flat, then $0 \to R' \otimes_R K \to R' \otimes_R L \to R' \otimes_R M \to 0$ is an exact sequence of R'-modules [B(a), p. 30, Proposition 4] and $R' \otimes_R M$ is R'-flat [B(a), p. 34, Corollary 2].

1.3 **Corollary.** *Let* $0 \to K \to L \to M \to 0$ *be an exact sequence of R-modules with L free and M flat. Then* $c(K)$ *is locally trivial.*

Proof. By localizing at P, it suffices to prove the corollary when R is quasi-local with maximal ideal P. If $c(K) \neq R$, then $K \subset PL$; hence $c(K) = 0$ by the following lemma:

1.4 **Lemma** [B(a), p. 66, Exercise 23-d]. *Under the hypothesis of 1.3, if* $J =$ *Jacobson radical of* R *and* $K \subset JL$, *then* $K = 0$.

Proof. For any $x \in K$, $x \in c(x)L \cap K = c(x)K$, the equality following from the flatness. Therefore $x \in c(x)JL$, and hence $c(x) = c(x)J$. By Nakayama's lemma, $c(x) = 0$, which implies $x = 0$. Q.E.D.

Note that when R is a domain the conclusion of 1.3 is equivalent to the assertion that either $K = 0$ or $c(K) = R$. Also, for arbitrary R, since $c(K)_P = c(K_P)$, if $K_P \neq 0$ for every prime P of R, then $c(K) = R$.

1.5 Theorem. *Let C be a polynomial ring over R (in any number of indeterminates), and let I be an ideal of C. If I is locally principal at primes of $R[X]$ and $c(I)$ is locally trivial, then C/I is a flat R-module.*

Proof. Fix a maximal ideal M' of C, and let $M = M' \cap R$. By 1.2 (d) and [B(a), p. 112, Corollary 1] it suffices to prove the following: If $I_{M'}$ is principal and $c(I)_M = 0$ or R_M, then for any $x \in I$, $x/1 \in (c(x) \cdot I)_{M'}$. Note that $c(I)_M = c(I_M)$ and $(c(x) \cdot I)_{M'} = (c(x)_M \cdot I_M)_{M'}$; and thus, by first localizing at M, we may assume R is quasi-local with maximal ideal M in proving this.

If $c(I) = 0$, then $I = 0$ and the assertion is immediate. Therefore assume $c(I) = R$. $I_{M'} = (f/1)C_{M'}$ for some $f \in I$. For any $x \in I$, $(x/1) = (f/1)(g/s)$, $g \in C$, $s \in C \setminus M'$. Therefore there exist $s_1, s_2 \in C \setminus M'$ such that $s_1 x = fgs_2$. But $s_i \in C \setminus M' \Rightarrow c(s_i) = R$ since R is quasi-local with maximal ideal M. Therefore by the content formula, $c(x) = c(fg)$. Then $R = c(I) \subset c(f)$ implies $c(f) = R$. It follows by the content formula that $c(x) = c(g)$. Thus $x/1 = (f/1)(g/s) = (fg/s) \in (c(g) \cdot I)_{M'} = (c(x) \cdot I)_{M'}$. Q.E.D.

The following result together with Corollary 1.3 will provide a converse to Theorem 1.5 in the case that C is a polynomial ring in one variable and I is locally finitely generated at primes of C.

1.6 Proposition. *Let I be an ideal of $R[X]$ such that $R[X]/I$ is R-flat. If I is locally finitely generated at primes of $R[X]$,[2] then I is locally principal at primes of $R[X]$ and $c(I)$ is locally trivial.*

Proof. Let P' be a prime of $R[X]$, and let $P = P' \cap R$. By localizing at P, we may assume that R is quasi-local with maximal ideal P. $R[X]/PR[X] \cong (R/P)[X]$ is a principal ideal domain; and hence there exists $f \in I$ such that $I = fR[X] + (PR[X] \cap I)$. But $R[X]/I$ is R-flat implies $PR[X] \cap I = PR[X] \cdot I$. Therefore $I_{P'} = fR[X]_{P'} + PR[X]_{P'} \cdot I_{P'}$. Since $I_{P'}$ is finitely generated, we conclude by Nakayama's lemma that $I_{P'} = fR[X]_{P'}$. Moreover, by Corollary 1.3, if $I \neq 0$, then $c(I) = R$. Q.E.D.

---

[2] Heinzer and Ohm, *The finiteness of I when $R[X]/I$ is R-flat. II*, Proc. Amer. Math. Soc. (to appear), have recently proved that $R[X]/I$ is R-flat implies I is locally finitely generated at primes of $R[X]$.

1.7 The condition that min I **contain a regular element.** Recall that for a non-zero ideal I of $R[X]$, min I denotes the set of nonzero elements of I of least degree. By the content formula, $f \in R[X]$ is regular if and only if $rc(f) = 0$ for $r \in R$ implies $r = 0$. In particular, if $f \in $ min I, then f is regular if and only if the leading coefficient of f is regular if and only if some coefficient of f is regular. If $R[X]/I$ is R-flat, we have seen in 1.3 that $c(I)$ is locally trivial; so if in addition min I contains a regular element, then it follows that $c(I) = R$.

2. I **is finitely generated.** Let R be a ring, let X be a single indeterminate, and let I be an ideal of $R[X]$ such that min I contains a regular element. The proof that $R[X]/I$ *is R-flat implies I is a finitely generated ideal* proceeds as follows. First prove the theorem in the case that R is quasi-local integrally closed with infinite residue field; then remove the infinite residue field assumption by adjoining an indeterminate (Theorem 2.7); next remove the quasi-local assumption (Corollary 2.16); and finally, remove the assumption that R is integrally closed. That part of the proof which concerns an integrally closed R uses only the hypothesis that $R[X]/I$ is R-torsion free and $c(I) = R$, rather than the apparently stronger hypothesis that $R[X]/I$ is R-flat. This is explained by Corollary 2.13, which asserts that these are equivalent statements for an integrally closed R.

Conditions for I to be principal will be discussed thoroughly in §3. For the present, we only need the following lemma.

2.1 Lemma. *Let I be an ideal of $R[X]$. Then the following are equivalent*:

(i) *There exists $f \in$ min I such that $c(f) = R$.*

(ii) $I = fR[X]$, *the leading coefficient of f is regular in R, and $c(I) = R$.*

(iii) *There exists $g \in$ min I such that $c(g) = aR$, a regular in R, and $R[X]/I$ is R-torsion-free.*

Moreover, when these hold, then $R[X]/I$ is R-flat and, for any $g \in$ min I, $c(g)$ is a principal ideal.

Proof. (i) $\Longrightarrow$ (ii): Let f be as in (i), and let d be the leading coefficient of f. Then d is regular in R. For $bd = 0$ for some $b \in R$ implies $bf = 0$ or deg $bf <$ deg f. Since $bf \in I$ and $f \in$ min I, it follows that $bf = 0$. But $c(f) = R$, so $b = 0$.

Now let $g \in I$. By the division algorithm [**ZS**(a), p. 30, Theorem 9], $d^n g = hf$ for some integer n and some $h \in R[X]$. Therefore, $d^n g \in c(h) \cdot fR[X]$; and since $c(h) = d^n c(g)$ by the content formula, $d^n g \in d^n c(g) fR[X]$. But d is regular in R and hence in $R[X]$, so then $g \in c(g) fR[X]$. Therefore $I = fR[X]$.

(ii) $\Longrightarrow$ (iii): $R[X]/I$ is R-flat by Theorem 1.5 and hence is also R-torsion-free [**B**(a), p. 29, Proposition 3].

(iii) $\Rightarrow$ (i): $c(g) = aR$ implies $g = af$ for some $f \in R[X]$. $R[X]/I$ is R-torsion-free and $g \in I$ implies $f \in I$. Therefore, $g \in \min I$ implies $f \in \min I$. Also, $ac(f) = c(g) = aR$ implies $c(f) = R$.

We have already seen in the course of the proof that (ii) implies $R[X]/I$ is R-flat. (ii) also implies for any $g \in \min I$, $g = af$, $a \in R$; and then $c(g) = aR$ since $c(f) = R$.

2.2 Remark. If D is a Bezout domain (i.e. finitely generated ideals are principal) and $D[X]/I$ is D-flat, then 2.1(iii) is satisfied, and hence, by 2.1(ii), I is principal. This observation includes the one variable case of Nagata's theorem discussed in §5. The case of a Prüfer domain D is disposed of almost as easily; then $D[X]/I$ is D-flat implies I is finitely generated, which will follow from our Theorem 2.15. However, I need not be principal in this case, as can be seen from Example 2.14(3).

We impose on I in much of what follows the condition that $\min I$ contain a regular element. This condition insures that $\deg \min I$ does not decrease when I is extended to a torsion-free overring of R, as the following lemma shows. Recall that $T(R)$ denotes the total quotient ring of R.

2.3 Lemma. *Let I be an ideal of $R[X]$, let R' be a torsion-free R-algebra, and let $I' = IR'[X]$. If $\min I$ contains a regular element, then $\min I'$ contains a regular element and $\deg \min I = \deg \min I'$. If $R \subset R' \subset T(R)$ and there exists $f \in \min I$ such that $c_R(f)R' = R'$, then f is regular in $R[X]$ and $I' = fR'[X]$.*

Proof. For the first assertion, suppose g is a regular element of $\min I$. Then the leading coefficient a of g is regular in R. If $g' \in I'$, $g' = \Sigma\, r_i g_i$, $r_i \in R'$, $g_i \in I$. By the division algorithm [ZS(a), p. 30, Theorem 9], $a^n g_i \in gR[X]$ for some n; so $a^n g' = hg$ for some $h \in R'[X]$. Thus $\deg g' \geq \deg g$ since the leading coefficient of g is regular in R and R' is R-torsion-free. It then also follows that the image of g in $R'[X]$ is a regular element of $\min I'$.

For the second assertion, observe that $R' \subset T(R)$ and $c_R(f) \cdot R' = R'$ implies $c_R(f)$ contains a regular element. It follows that the annihilator of the coefficients of f is zero and hence that f is regular in $R[X]$. Therefore, $\deg f = \deg \min I = \deg \min I'$, and hence $f \in \min I'$. But $c_{R'}(f) = c_R(f) \cdot R' = R'$, so $I' = fR'[X]$ by Lemma 2.1.   Q.E.D.

Note that it can easily happen that $R[X]/I$ is R-flat and yet $\min I$ does not contain a regular element. For an example (with $I \neq 0$), let R be any ring which contains a nonnilpotent zero-divisor a, and let $I = (1 + aX)$. Then a simple computation shows $I \neq R[X]$. By Theorem 1.5, $R[X]/I$ is R-flat and hence a fortiori is R-torsion-free. But then $I \cap R = \min I \cup \{0\}$ consists only of zero-divisors.

2.4 Lemma. *Let I be an ideal of $R[X]$, and let R' be a ring such that*

$R \subset R' \subset T(R)$. If $R[X]/I$ is R-torsion-free, then $IR'[X] \cap R[X] = I$.

Proof. Let $f \in IR'[X] \cap R[X]$. Then $f = \Sigma (a_i/s_i)f_i$ with $f_i \in I$, a_i, $s_i \in R$, and s_i regular for every i. Then $sf = \Sigma a_i' f_i \in I$ for some regular element s of R. But since $R[X]/I$ is R-torsion-free, then $f \in I$.

2.5 Lemma. Let $b \in R$, and let ϕ denote the R-automorphism of $R[X]$ defined by $\phi(X) = X + b$. Then $c(f) = c(\phi(f))$ for each $f \in R[X]$.

Proof. Let $a_n X^n + \cdots + a_0 = f(X) \in R[X]$. Then

$$\phi(f(X)) = a_n(X + b)^n + a_{n-1}(X + b)^{n-1} + \cdots + a_0$$

$$= a_n\left(\sum_{i=0}^{n}\binom{n}{i}X^{n-i}b^i\right) + a_{n-1}\left(\sum_{i=0}^{n-1}\binom{n-1}{i}X^{n-1-i}b^i\right) + \cdots + a_1(X + b) + a_0$$

$$= a_n X^n + \left(a_n\binom{n}{1}b + a_{n-1}\right)X^{n-1} + \cdots$$

$$+ \left(a_n\binom{n}{n-1}b^{n-1} + a_{n-1}\binom{n-1}{n-2}b^{n-2} + \cdots + a_1\right)X$$

$$+ (a_n b^n + a_{n-1}b^{n-1} + \cdots + a_0).$$

2.6 Lemma. Let R' be an R-algebra, and let $g, h \in R'[X]$ be such that $g(0) = h(0) = 1$. If the coefficients of gh are integral over R, then the coefficients of g and h are integral over R.

Proof. Let $g(X) = a_n X^n + \cdots + a_1 X + 1$, and let $h(X) = b_m X^m + \cdots + b_1 X + 1$. Let $g^* = a_n + a_{n-1}X + \cdots + a_1 X^{n-1} + X^n$ and similarly $h^* = b_m + b_{m-1}X + \cdots + b_1 X^{m-1} + X^m$. Then $g^* h^*$ has the same coefficients as gh, and hence the coefficients of $g^* h^*$ are integral over R. But then the coefficients of g^* and h^* are integral over R by [B(c)], p. 17, Proposition 11]. Q.E.D.

The proof of the following theorem will be broken into two parts. The case that R/m is finite requires Lemmas 2.9 and 2.10 and will be given after Lemma 2.10.

2.7 Theorem. Let R, m be a quasi-local ring, and let R' be a ring such that $R \subset R' \subset T(R)$ and R is integrally closed in R'. If I is an ideal of $R[X]$ such that $R[X]/I$ is R-torsion-free and $c(I) = R$, and if $c_R(g)R' = R'$ for some $g \in \min I$, then I is principal and is generated by a regular element of $\min I$.

Proof (in the case that R/m is infinite). Let $f \in I$ be such that $c(f) = R$.

Since R/m is infinite there exists $a \in R$ such that $f(a) \not\equiv 0 \pmod{m}$; and hence $f(a)$ is a unit of R. Let ϕ be the R-automorphism of $R[X]$ defined by $\phi(X) = X + a$. Then since h and $\phi(h)$ have the same content (by Lemma 2.5) and the same degree for each $h \in R[X]$ and since $\phi(f)(0) = f(a)$, we may, after replacing I with $\phi(I)$, assume that $f(0)$ is a unit of R. After dividing f by $f(0)$, we may further assume $f(0) = 1$. By Lemma 2.3, $IR'[X] = gR'[X]$. In particular, $f = gh$ for some $h \in R'[X]$. Thus, $g(0)$ is a unit of R'; and hence in $R'[X]$ we have $f = (g/g(0)) \cdot (g(0) \cdot h)$. Therefore, by Lemma 2.6, the coefficients of $g/g(0)$ are integral over R. Since R is integrally closed in R', it follows that $g/g(0) \in R[X]$; and hence $g/g(0) \in R[X] \cap IR'[X] = I$, the equality being a consequence of Lemma 2.4. But then $g/g(0) \in \min I$ and $c_R(g/g(0)) = R$, so the result follows from Lemma 2.1.

2.8 Definition of $R(Y)$. In removing from 2.7 the restriction that R/m be infinite, we pass to a certain overring $R(Y)$ of R which is defined as follows [$\mathbf{Na}_3$, p. 18]. Let Y be an indeterminate, and let $S = \{f \in R[Y] \mid c(f) = R\}$. S is a multiplicative system of $R[Y]$ consisting of regular elements, and the ring $R[Y]_S$ is customarily denoted $R(Y)$. We list a few properties of this ring.

(i) An ideal m' of $R(Y)$ is maximal if and only if there exists a maximal ideal m of R such that $m' = mR(Y)$ [$\mathbf{Na}_3$, p. 18].

(ii) For each ideal A of R, $R(Y)/AR(Y) \cong (R/A)(Y)$ [$\mathbf{Na}_3$, p. 18].

(iii) $R(Y)$ is R-flat [$\mathbf{Na}_3$, p. 64, Exercise 2]; and hence by (i) $R(Y)$ is a faithfully flat R-module.

(iv) $R(Y)[X]$ is a faithfully flat $R[X]$-module [$\mathbf{B}(a)$, p. 48, Proposition 5].

2.9 Lemma. *Let R, m be a quasi-local ring, and let I be an ideal of $R[X]$. If $IR(Y)[X] = fR(Y)[X]$ for some regular element $f \in \min(IR(Y)[X])$, then I is principal and is generated by a regular element of $\min I$.*

Proof. Since $R(Y)[X]$ is a faithfully flat $R[X]$-module, every ideal of $R[X]$ extends and contracts to itself [$\mathbf{B}(a)$, p. 51, Proposition 9]. Therefore it suffices to show that there exists $g \in I$ such that $\deg_X g = \deg_X f$ and $IR(Y)[X] = gR(Y)[X]$. We may assume $f \in IR[Y, X]$. Then $f = f_0 + f_1 Y + \cdots + f_t Y^t$, $f_i \in I$; and hence $\deg_X f \geq \deg_X f_i$. But $f \in \min(IR(Y)[X])$ and $f_i \in IR(Y)[X]$, so $\deg_X f \leq \deg_X f_i$. Thus $\deg_X f_i = \deg_X f$. Since the leading coefficient of f is regular in $R(Y)$ and $IR(Y)[X] = fR(Y)[X]$, we have $f_i = a_i f$ with $a_i \in R(Y)$. Thus $f = (a_0 + a_1 Y + \cdots + a_t Y^t)f$; and since f is regular in $R(Y)[X]$, $1 = a_0 + a_1 Y + \cdots + a_t Y^t$. Hence $a_j \notin mR(Y)$ for some j. Then a_j is a unit of $R(Y)$, since $mR(Y)$ is the unique maximal ideal of $R(Y)$. Therefore $f_j R(Y)[X] = fR(Y)[X]$.   Q.E.D.

Example 2.14(3) will show that 2.9 is no longer valid without the assumption that R is quasi-local.

2.10 Lemma. *Let I be an ideal of $R[X]$ such that* min I *contains a regular element of $R[X]$. If $R[X]/I$ is R-torsion-free then $R(Y)[X]/IR(Y)[X]$ is $R(Y)$-torsion-free.*

Proof. Suppose $af \in IR(Y)[X]$, a regular in $R(Y)$ and $f \in R(Y)[X]$. Then $a = a'/s_1$, $f = f'/s_2$, where a' is regular in $R[Y]$, $f' \in R[Y, X]$, $s_i \in S = \{s \in R[Y] |$ $c(s) = R\}$. There exists $s \in S$ such that $sa'f' \in IR[Y, X]$, and thus it suffices to show $R[Y, X]/IR[Y, X]$ is $R[Y]$-torsion-free.

Let T be the total quotient ring of R, and let g be a regular element of min I. Since the leading coefficient of g is regular in R, we have $c_T(g) = T$ and $IT[X] = gT[X]$ by Lemma 2.3. Hence by Theorem 1.5, $T[X]/IT[X]$ is T-flat. Therefore, $T[Y, X]/IT[Y, X]$ ($\cong T[X]/IT[X] \otimes_T T[Y]$) is $T[Y]$-flat by [B(a), p. 34, Corollary 2] and a fortiori $T[Y]$-torsion-free. Now let $a \in R[Y]$ be regular, let $f \in R[Y, X]$, and suppose $af \in IR[Y, X] \subset IT[Y, X]$. Then a is regular in $T[Y]$; so $f \in IT[Y, X]$. Thus, there exists a regular element r of R such that $rf \in IR[Y, X]$. Regarding f as a polynomial in Y, we then have $r \cdot c_{R[X]}(f) = c_{R[X]}(rf) \subset I$. Therefore, $c_{R[X]}(f) \subset I$ since $R[X]/I$ is R-torsion-free, and hence $f \in IR[Y, X]$. Q.E.D.

When R is noetherian, the conclusion of Lemma 2.10 remains valid without the assumption that min I contains a regular element (a related statement can be found in [Na$_3$, p. 63, Theorem 18.12]). It is thus perhaps of some interest to digress for a moment in order to consider the possibility of omitting this assumption from 2.10. Let a be a regular element of $R[Y]$, so that $(0 : c_R(a))_R = 0$; and suppose $af \in IR[Y, X]$ for some $f \in R[Y, X]$. Now regard af as a polynomial in Y and apply the content formula to obtain $c_R(a)^n c_{R[X]}(f) \subset I$ for some $n \geq 1$. If $c_R(a)$ contains a regular element, then we can conclude that $c_{R[X]}(f) \subset I$ (assuming $R[X]/I$ is R-torsion-free), and hence $f \in IR[Y, X]$. Thus, if R is a ring with the property that, for any finitely generated ideal A of R, $(0 : A)_R = 0$ implies A contains a regular element (e.g. a domain or a noetherian ring), then the conclusion of 2.10 remains valid without the assumption that min I contains a regular element.

The following example shows that this assumption cannot be omitted in general. By [Ka$_1$, p. 63, Exercise 7], there exists a ring R with the following properties: (i) R is its own total quotient ring, (ii) there exist nonzero p_1, $p_2 \in R$ such that (p_1, p_2) has zero annihilator, (iii) there exist nonzero m_1, $m_2 \in R$ such that $p_1 m_1 = 0$, $p_2 m_2 = 0$ and $m_2 \notin p_1 R$. Let $a = p_1 + p_2 Y$, let $g = p_1 + p_2 X$, and let $I = gR[X]$. By (ii), a is a regular element of $R[Y]$ and g is a regular element of $R[Y, X]$; and by (i) $R[X]/I$ is trivially R-torsion-free. Let $f = m_2 + m_1 X$. Then $af = p_1 m_2 + p_2 m_1 YX = (m_2 + m_1 Y)g \in IR[Y, X]$. However, $f \notin gR[Y, X]$ since $m_2 \notin p_1 R$. Thus, $R[Y, X]/IR[Y, X]$ is not $R[Y]$-torsion-free.

We shall now complete the proof of Theorem 2.7.

Proof of Theorem 2.7 in the general case. If R/m is infinite we merely apply the previous proof. Otherwise we must reduce to this case by passing to the ring $R(Y)$, which is quasi-local with infinite residue field by 2.8. By Lemma 2.10, $R(Y)[X]/IR(Y)[X]$ is $R(Y)$-torsion-free; and $c(I) = R$ implies $c_{R(Y)}(IR(Y)[X]) = R(Y)$. By Lemma 2.9 it suffices to show $IR(Y)[X]$ is principal generated by a regular element of $\min(IR(Y)[X])$. For this it suffices to find an overring R'' of $R(Y)$, contained in the total quotient ring of $R(Y)$, such that $R(Y)$ is integrally closed in R'' and such that $c_{R(Y)}(g)R'' = R''$. We claim $R'' = R'[Y]_S$ is such a ring, where $S = \{f \in R[Y] \mid c_R(f) = R\}$. Since $c_R(g)R' = R'$, it follows that $c_{R(Y)}(g)R'[Y]_S = R'[Y]_S$. Also, $R[Y]$ is integrally closed in $R'[Y]$ [**B(c)**, p. 19, Proposition 13], and hence $R(Y) = R[Y]_S$ is integrally closed in $R'[Y]_S$ [**B(c)**, p. 22, Proposition 16]. Finally, any element of $R'[Y]_S$ can be written in the form f/g with $f, g \in R[Y]$ and $c_R(g) = sR$, with s regular in R; and hence $R'[Y]_S$ is contained in the total quotient ring of $R[Y]$. Q.E.D.

The next corollary to Theorem 2.7 is the main result in §2 for the case that R is quasi-local and integrally closed (in its total quotient ring).

2.11 Corollary. *Let R, m be a quasi-local integrally closed ring, and let I be an ideal of $R[X]$ such that $\min I$ contains a regular element. If $R[X]/I$ is R-torsion-free and $c(I) = R$, then I is principal and is generated by a regular element of $\min I$.*

Proof. Take $T(R)$ to be the R' of Theorem 2.7. If g is a regular element of $\min I$, then the leading coefficient of g is regular in R and hence is a unit of $T(R)$. Q.E.D.

Another consequence of Theorem 2.7 is Corollary 2.13, which illuminates the relationship between $R[X]/I$ being R-flat and $R[X]/I$ being R-torsion-free. The following lemma will enable us to give the corollary in a global form.

2.12 Lemma. *Let I be an ideal of $R[X]$ such that $\min I$ contains a regular element. Then $R[X]/I$ is R-torsion-free (if and) only if $R_P[X]/IR_P[X]$ is R_P-torsion-free for every prime P of R.*

Proof. Let $T = T(R)$, let P be a prime of R, and suppose $(a/s)(f/s') \in IR_P[X]$ where $a \in R$, s, $s' \in R \backslash P$, $f \in R[X]$, and a/s is regular in R_P. Then $(a/s)(f/s') \in IR_P[X] \subset IT_P[X]$. But $T[X]/IT[X]$ is T-flat by Lemma 2.3 and Lemma 2.1, and hence $T_P[X]/IT_P[X]$ is T_P-flat and a fortiori T_P-torsion-free. Also T_P is R_P-flat, so a/s is regular in T_P, and hence $f/s' \in IT_P[X]$. Therefore there exists $s'' \in R \backslash P$ such that $s''f \in IT[X]$, and hence there exists a regular element $r \in R$ such that $rs''f \in I$. But since $R[X]/I$ is R-torsion-free, then $s''f \in I$; therefore $f/1 \in IR_P[X]$.

2.13 **Corollary.** *Let I be an ideal of* $R[X]$ *such that* $\min I$ *contains a regular element and consider the following properties*:

(i) *For every prime ideal* P *of* R, $IR_P[X]$ *is principal generated by an element of* $\min(IR_P[X])$; *and* $c(I) = R$.

(ii) *For every prime ideal* P *of* R, $IR_P[X]$ *is principal; and* $c(I) = R$.

(iii) $R[X]/I$ *is* R-*flat*.

(iv) $R[X]/I$ *is* R-*torsion-free, and* $c(I) = R$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv); *and if* R *is integrally closed, then* (iv) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Apply Theorem 1.5.

(iii) $\Rightarrow$ (iv): $c(I) = R$ by Corollary 1.3 and the fact that $c(I)$ contains a regular element.

(iv) $\Rightarrow$ (i): Let P be a prime ideal of R and let g be a regular element of $\min I$. Then $R_P \subset T(R)_P \subset T(R_P)$, and R_P is integrally closed in $T(R)_P$ by [B(c), p. 22, Proposition 16]. Also, by Lemma 2.3 the canonical image $g/1$ of g in $R_P[X]$ is in $\min(IR_P[X])$; and since $c_R(g)T(R) = T(R)$ then $T(R)_P = c_{R_P}(g/1)T(R)_P$. Since $R_P[X]/IR_P[X]$ is R_P-torsion-free by Lemma 2.12, (i) follows from Theorem 2.7.   Q.E.D.

Note that the assumption that $\min I$ contains a regular element is needed in Lemma 2.12. To see this it suffices to exhibit a ring R having a prime P consisting of zero-divisors and such that PR_P contains a regular element of R_P. For then $R[X]/PR[X]$ would be R-torsion-free but $R_P[X]/PR_P[X]$ would not be R_P-torsion-free. The existence of a ring R with such a prime can be seen as follows. Let R_0 be a noetherian ring of Krull dimension ≥ 3. Then there exists a chain of prime ideals $P_0 < P_1 < P_2 < P_3$ of R_0, and by [ZS(a), p. 230, Theorem 21] there exists an ideal A of R_0 whose associated primes are exactly P_1 and P_3. Letting $R = R_0/A$, we get a ring R whose associated primes of (0) are exactly $\bar{P}_1 = P_1/A$ and $\bar{P}_3 = P_3/A$. Thus $\bar{P}_2$ consists of zero-divisors since $\bar{P}_2 \subset \bar{P}_3$; but $\bar{P}_2 R_{\bar{P}_2}$ contains a regular element of $R_{\bar{P}_2}$, for otherwise $\bar{P}_2 R_{\bar{P}_2}$ would be an associated prime of (0) in $R_{\bar{P}_2}$ [B(b), p. 137, Proposition 8] and hence $\bar{P}_2$ would be an associated prime of (0) in R [B(b), p. 134, Proposition 5].

Before proceeding further, we shall give some examples to illustrate what can happen if one alters the hypothesis of Corollary 2.13.

2.14 **Examples.** (1) Let R be any ring for which $\bar{R}$ is different from its integral closure $\bar{R}$. Then there exists an ideal I of $R[X]$ such that $\min I$ contains a regular element and such that 2.13(iv) is satisfied but 2.13(iii) is not. To see this, choose $\xi \in \bar{R} \setminus R$ and let I be the kernel of the R-homomorphism $R[X] \to R[\xi]$. $R[\xi]$ is R-torsion-free since $R[\xi] \subset T(R)$, and $c(I) = R$ since ξ

is integral over R. Moreover, since $\xi = a/b$, $a, b \in R$, b regular, $bX - a$ is a regular element of min I. However, $R[\xi]$ is not R-flat since $R < R[\xi] \subset \overline{R}$ [A, p. 803, Corollary 2].

(2) [Na$_2$, p. 446, Example 2]. This is an example of a quasi-local domain D, which is not integrally closed, and an ideal I of $D[X]$ such that 2.13(iii) is satisfied but not 2.13(ii). Let k be a field and t be an indeterminate, let $D = k[t^2, t^3]_{(t^2, t^3)}$, and let I be the kernel of the D-homomorphism $D[X] \rightarrow D[1/t]$. $D[1/t]$ is the quotient field of D and hence is D-flat; but $t^3X - t^2 \in$ min I and does not have principal content, so I cannot be principal by 2.1.

(3) This is an example of an integrally closed domain D, which is not quasi-local, and an ideal I of $D[X]$ such that 2.13(iii) is satisfied but I is not principal. (Corollary 2.13 shows (iii) implies I is principal when R is quasi-local and integrally closed.) Let D be a Prüfer domain which is not Bezout (e.g. any Dedekind domain with nontrivial class group [C, p. 222, Theorem 7]). Then there exists an ideal (a, b) of D which is not principal. Since $D[b/a]$ is contained in the quotient field of D, $D[b/a]$ is D-torsion-free and hence by [B(a), p. 29, Proposition 3] is D-flat. Therefore, if I is the kernel of the D-homomorphism $D[X] \rightarrow D[b/a]$, then $c(I) = D$ by Corollary 1.3. Moreover, I is not principal. For, if $I = fD[X]$, then $c(I) = D$ implies $c(f) = D$. But $aX - b \in I$ implies $aX - b = df$, $d \in D$; and then by equating contents, $(a, b) = (d)$, a contradiction to the choice of (a, b). (Nagata [Na$_2$, p. 446, Example 1] gives a similar example.)

Note that Example 3 also shows that the quasi-local assumption is needed in Lemma 2.9. For $D[X]/I$ is D-flat implies (by tensoring; see 1.2) that $D(Y)[X]/ID(Y)[X]$ is $D(Y)$-flat; and $D(Y)$ is the Kronecker function ring of D [G, p. 384, Theorem 27.4], which is Bezout [Kr, p. 559, Satz 14], so $ID(Y)[X]$ is principal by 2.2.

The next theorem gives some further conditions which are equivalent to 2.13(i) and is the tool needed to pass from the local to the global case.

2.15 Theorem. *Let I be an ideal of $R[X]$ such that* min I *contains a regular element, and assume* $c(I) = R$. *Then the following are equivalent*:

(i) *For every prime ideal P of R, $IR_P[X]$ is principal generated by an element of* $\min(IR_P[X])$.

(ii) $c(\min I) = R$.

(iii) $I = (f_1, \cdots, f_n)$, $f_i \in$ min I.

Proof. (i) $\Rightarrow$ (ii): Since by Lemma 2.3, deg min $I = \deg \min(I_P)$ for each prime P of R, it follows that for each prime P of R there exists $f \in$ min I such that $I_P = fR_P[X]$. Then $c(f)_P = R_P$, and hence $c(f) \not\subset P$. It follows that $c(\min I) = R$.

(ii) $\Rightarrow$ (iii): By (ii) there exist $f_1, \cdots, f_n \in$ min I such that $c(f_1, \cdots, f_n) = R$. Therefore for any prime P of R, $c(f_i) \not\subset P$ for some i; and since the image of f_i in $R_P[X]$ is in min (I_P), it follows from Lemma 2.1 that $I_P = f_i R_P[X]$. This shows that $I_P = (f_1, \cdots, f_n)_P$ for each prime P of R, and hence $I = (f_1, \cdots, f_n)$ [B(a), p. 111, Theorem 1].

(iii) $\Rightarrow$ (i): Since $c(f_1, \cdots, f_n) = R$, for any prime P of R there exists f_i such that $c(f_i) \not\subset P$. But since the image of f_i in $R_P[X]$ is in min (I_P) (by Lemma 2.3), it follows from 2.1 that $I_P = f_i R_P[X]$.

2.16 Corollary. *Let R be integrally closed, and let I be an ideal of $R[X]$ such that min I contains a regular element. If $R[X]/I$ is R-flat, then I is generated by finitely many elements of min I and $c(I) = R$.*

Proof. That $c(I) = R$ follows from Corollary 1.3 and the assumption that min I contains a regular element. Corollary 2.13 shows that 2.15(i) is satisfied, and then the remaining conclusion is just 2.15(iii).   Q.E.D.

In 2.21 we shall give an example of an integrally closed R such that $R[X]/I$ is R-flat, $c(I) = R$, and 2.15(i) is satisfied but such that I is not finitely generated. Thus, the assumption in 2.15 and 2.16 that min I contains a regular element is essential. Note also that Example 2.14(3) shows that under the hypotheses of Corollary 2.16, I need not be principal; however, if one imposes the additional condition that the maximal spectrum of R be a noetherian space of dimension zero, then I may be seen to be principal as follows: Let g be a regular element of min I. By Corollary 2.13 and Lemma 2.1, it follows that $c(g)_P$ is principal for every prime P of R. But then $c(g)$ is principal by [S, p. 318, Theorem 1], and hence by Lemma 2.1, I is principal.

The final part of the proof of the main theorem of §2 consists in going from $\overline{R}$ to R, where $\overline{R}$ denotes the integral closure of R in its total quotient ring $T(R)$.

2.17 Lemma. *Let R, m be a quasi-local ring, let I be an ideal of $R[X]$, let $g \in I$, and let P' be a prime ideal of $R[X]$ lying over m. If $c(I) = R$ and $I\overline{R}[X]_{P''} = g\overline{R}[X]_{P''}$ for every prime P'' of $\overline{R}[X]$ lying over m, then $IR[X]_{P'} = gR[X]_{P'}$.*

Proof. First we show $I\overline{R}[X]_{P'} = g\overline{R}[X]_{P'}$, and by [B(a), p. 111, Theorem 1] it suffices to check this locally at the maximal ideals of $\overline{R}[X]_{P'}$. It follows from the integralness of $\overline{R}[X]_{P'}$ over $R[X]_{P'}$ that any maximal ideal of $\overline{R}[X]_{P'}$ contracts to P' in $R[X]$. Therefore the localizations of $\overline{R}[X]_{P'}$ at its maximal ideals are all of the form $\overline{R}[X]_{P''}$, where P'' is a prime of $\overline{R}[X]$ which contracts to P' in $R[X]$ and hence to m in R. Since $I\overline{R}[X]_{P''} = g\overline{R}[X]_{P''}$ for all such P'' by hypothesis, we have $I\overline{R}[X]_{P'} = g\overline{R}[X]_{P'}$.

Now we show $IR[X]_{P'} = gR[X]_{P'}$. Since $I\overline{R}[X]_{P'} = g\overline{R}[X]_{P'}$, for any $f \in I$

there exist $h \in \overline{R}[X]$ and $s_1, s_2 \in R[X] \setminus P'$ such that $s_1 f = s_2 gh$. Moreover, $P' \cap R = m$ implies $c_R(s_i) = R$. Therefore, $c_R(f) = c_R(gh) \subset c_{\overline{R}}(g)$. Since $c(I) = R$, there exists $f_1 \in I$ such that $c_R(f_1) = R$; so the above reasoning applied to f_1 yields $R = c_R(f_1) \subset c_{\overline{R}}(g)$. Hence by the lying-over theorem, $c_R(g) = R$. Therefore, $c_R(f) = c_R(gh) = c_R(h)$, so $c_R(h) \subset R$ and $h \in R[X]$. Thus, $f \in gR[X]_{P'}$. Q.E.D.

2.18 Theorem. *Let I be an ideal of $R[X]$ such that* $\min I$ *contains a regular element. Then the following are equivalent*:

  (i) $R[X]/I$ *is R-flat.*

  (ii) $\overline{R}[X]/I\overline{R}[X]$ *is $\overline{R}$-flat.*

  (iii) $I_{P'}$ *is principal for every prime P' of $R[X]$, and $c(I) = R$.*

Proof. (i) $\Rightarrow$ (ii): Since $R[X]/I$ is R-flat, the sequence $0 \to I \otimes_R \overline{R} \to R[X] \otimes_R \overline{R} \to (R[X]/I) \otimes_R \overline{R} \to 0$ is exact; and hence $(R[X]/I) \otimes_R \overline{R} \cong \overline{R}[X]/I\overline{R}[X]$ (as $\overline{R}$-algebras). But $(R[X]/I) \otimes_R \overline{R}$ is $\overline{R}$-flat by [**B**(a), p. 34, Corollary 2].

(ii) $\Rightarrow$ (iii): Let P' be a prime ideal of $R[X]$, and let $P = P' \cap R$. By localizing at P, we may assume R is quasi-local with maximal ideal P. Then $(R/P)[X]$ is a principal ideal domain, so there exists $g \in I$ such that $I = gR[X] + (PR[X] \cap I)$. Therefore, $I\overline{R}[X] = g\overline{R}[X] + (P\overline{R}[X] \cap I\overline{R}[X])$. Since $\overline{R}[X]/I\overline{R}[X]$ is $\overline{R}$-flat, $P\overline{R}[X] \cap I\overline{R}[X] = P\overline{R}[X] \cdot I\overline{R}[X]$ [**B**(a), p. 33, Corollary]. Therefore, $I\overline{R}[X] = g\overline{R}[X] + P\overline{R}[X] \cdot I\overline{R}[X]$. But $\min(I\overline{R}[X])$ contains a regular element by Lemma 2.3, so $I\overline{R}[X]$ is a finitely generated ideal by Corollary 2.16. Thus, by Nakayama's lemma, $I\overline{R}[X]_{P''} = g\overline{R}[X]_{P''}$ for every prime P'' of $\overline{R}[X]$ lying over P. We can now apply Lemma 2.17 to get $IR[X]_{P'} = gR[X]_{P'}$ provided we first note that $\overline{R}[X]/I\overline{R}[X]$ is $\overline{R}$-flat implies $c(I\overline{R}[X]) = \overline{R}$ by Corollary 1.3 and hence that $c(I) = R$.

(iii) $\Rightarrow$ (i): Apply Theorem 1.5. Q.E.D.

Note that (ii) $\Rightarrow$ (iii) of the above argument actually proves that for each prime P of R there exists $g \in I$ such that $I_{P'} = gR[X]_{P'}$ for every prime P' of $R[X]$ lying over P. Thus, if (ii) is valid and α is the cardinality of the set of primes P of R which are contractions of maximal ideals of $R[X]$ (called the G-ideals of R by Kaplansky [**Ka**$_1$, p. 17, Theorem 27]), then I can be generated by α elements.

2.19. Main theorem of §2. *Let I be an ideal of $R[X]$ such that* $\min I$ *contains a regular element. If $R[X]/I$ is a flat R-module, then I is a finitely generated ideal.*

Proof. As observed previously, $R[X]/I$ is R-flat implies $\overline{R}[X]/I\overline{R}[X]$ is $\overline{R}$-flat; and $\min I$ contains a regular element implies $\min(I\overline{R}[X])$ does also by Lemma 2.3. Therefore, $I\overline{R}[X]$ is finitely generated by Corollary 2.16. Hence there exists a finitely generated ideal A of $R[X]$ such that $A \subset I$ and $A\overline{R}[X] = I\overline{R}[X]$.

Claim. $A = I$. It suffices to show $A_{P'} = I_{P'}$ for every prime P' of $R[X]$. By localizing at $P = P' \cap R$, we may assume R is quasi-local with maximal ideal P. Let $\overline{P}$ be a prime ideal of $\overline{R}$ lying over P, let $\overline{\phi}$ be the homomorphism $\overline{R}[X] \to (\overline{R}/\overline{P})[X]$ and let ϕ be the homomorphism $R[X] \to (R/P)[X]$. One checks easily (or applies [B(a), p. 48, Proposition 5]) that since R/P is a field, every ideal of $(R/P)[X]$ extends and contracts to itself in $(\overline{R}/\overline{P})[X]$. Therefore, $\overline{\phi}(A\overline{R}[X]) = \overline{\phi}(I\overline{R}[X])$ implies $\phi(A) = \phi(I)$. Thus, $I = A + (PR[X] \cap I)$. Since $R[X]/I$ is R-flat, $PR[X] \cap I = PR[X] \cdot I$ [B(a), p. 33, Corollary]; so $I = A + PR[X] \cdot I$. Then $I_{P'} = A_{P'} + PR[X]_{P'} \cdot I_{P'}$; and since $I_{P'}$ is finitely generated by Theorem 2.18, we conclude by Nakayama's lemma that $I_{P'} = A_{P'}$.   Q.E.D.

Note that the last part of the above proof actually shows that (under the hypotheses of 2.19) for any ideal A of $R[X]$ such that $A \subset I$, $A\overline{R}[X] = I\overline{R}[X]$ implies $A = I$. This portion of the proof could also be done using (ii) $\Leftrightarrow$ (i) of 2.18 and Lemma 2.4.

2.20 Corollary. *Let I be an ideal of $R[X]$ such that* min I *contains a regular element. Then $R[X]/I$ is R-flat if and only if I is an invertible ideal and $c(I) = R$.*

Proof. ($\Rightarrow$) I is finitely generated by Theorem 2.19, and $I_{P'}$ is principal for every prime P' of $R[X]$ by Theorem 2.18. Thus, since I contains a regular element, I is invertible by [B(a), p. 148, Theorem 4]. That $c(I) = R$ follows from Corollary 1.3 and the fact that $c(I)$ contains a regular element.

($\Leftarrow$) If I is invertible, then I is locally principal at primes of $R[X]$ by [B(a), p. 148, Theorem 4]. Therefore $R[X]/I$ is R-flat by Theorem 1.5.   Q.E.D.

Let us consider for a moment the special case that min I contains a regular element of degree 1. Then $R[X]/I = R[\xi]$ where $\xi \in T(R)$. In this case we can give a simple construction for the generators of I. Let $\xi = x/y$, x, $y \in R$, y regular. By 1.2, $(yR : x)R[x/y] = (yR[x/y] : x)$ and hence $1 \in (yR : x)R[x/y]$. Therefore there exist $a_i \in (yR : x)$ such that $f = 1 + a_0 + a_1 X + \cdots + a_n X^n \in I$. Let $b_i \in R$ be such that $a_i x = y b_i$. Then it follows that $I = (f, a_0 X - b_0, \cdots, a_n X - b_n)$. For, if P' is a prime ideal of $R[X]$ containing $(f, a_0 X - b_0, \cdots, a_n X - b_n)$, then $f \in P'$ implies $a_i \notin P'$ for some i. Therefore, if $P = P' \cap R$, then $IR_P[X] = (a_i X - b_i)R_P[X]$ by Lemma 2.1, and then a fortiori $IR[X]_{P'} = (f, a_0 X - b_0, \cdots, a_n X - b_n)R[X]_{P'}$.

We conclude this section with an example which shows that 2.19 and many of our other results are not true without the hypothesis that min I contains a regular element. Recall that a ring R is said to be absolutely flat if every R-module is flat [B(a), p. 64, Exercises 16–17; p. 168, Exercise 9; p. 173, Exercise 17].

2.21 Example. This is an example of a ring R and an ideal I of $R[X]$ such

that $R[X]/I$ is R-flat but yet I is not a finitely generated ideal.

Let A be any non-finitely-generated ideal of an absolutely flat ring R. (For example, take R to be a countable direct product of copies of a field and A to be the direct sum ideal.) Then $I = AR[X]$ is also not finitely generated, and $R[X]/I$ is R-flat because every R-module is. For later reference, note that if A is chosen to be R-projective (as is the case with the direct sum ideal), then $AR[X]$ is $R[X]$ projective.

A slight modification of this example, namely $I = AR[X] + XR[X]$, gives an example with the additional property that I contains a regular element and $c(I) = R$. In this case $R[X]/I \cong R/A$ is even a cyclic flat R-module.

Added January 28, 1972. A conversation with Vasconcelos has brought our attention to the recent important work of M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, Invent. Math. 13(1971), 1–89. Corollary 3.4.7 of that paper is the n-variable generalization of our Theorem 2.19 in the case that R is a domain, and Corollary 3.4.2 contains the n-variable generalization of the theorem previously attributed to Heinzer and Ohm in footnote 2.

3. I principal. Consider for the moment the case of a *quasi-local* ring R and an ideal I of $R[X]$ such that min I contains a regular element. The problem treated in this section is to find conditions on $R[X]/I$ which are equivalent to I being principal and of content R. We know by Lemma 2.1 that I is principal and of content R implies $R[X]/I$ is R-flat, and we have proved in Corollary 2.11 that the converse is true when R is integrally closed. However, Example 2.14(2) shows that the converse is no longer valid when the integral closure assumption is dropped. Thus, the main problem is to determine what conditions in addition to $R[X]/I$ being R-flat must be imposed on $R[X]/I$ in order to conclude that I is principal. The Summary 3.6 provides a satisfying solution to this problem (when min I contains a regular element).

ξ will denote the equivalence class of X in $R[X]/I$. Then $R[X]/I = \phi(R)[\xi]$, where $\phi: R[X] \to R[X]/I$ is the canonical homomorphism. Regarding $\phi(R)[\xi]$ as an R-algebra via the homomorphism $R \to \phi(R)$, we can more simply use the notation $R[\xi]$ to denote the ring $\phi(R)[\xi]$. If ξ is a regular element of $\phi(R)[\xi]$, then $1/\xi$ is in $T(\phi(R)[\xi])$, so $\phi(R)[1/\xi]$ makes sense; and we again write $R[1/\xi]$ when no confusion can arise.

We remind the reader that $R\langle \alpha_1, \cdots, \alpha_n \rangle$ denotes the R-module having generators $\alpha_1, \cdots, \alpha_n$. When the ring R is understood, we shall also merely use $\langle \alpha_1, \cdots, \alpha_n \rangle$ for this R-module.

3.1 Lemma. *Let R, m be a quasi-local ring, let I be an ideal of $R[X]$, and let ξ denote the equivalence class of X in $R[X]/I$. If there exists $f = a_0 + a_1 X + \cdots + a_n X^n \in I$ with $a_0, \cdots, a_{l-1} \in m$ and $a_l = 1$ $(l \geq 0)$, then $\langle 1, \xi, \cdots, \xi^{n+i} \rangle = \langle 1, \xi, \cdots, \xi^{l-1}, \xi^{l+1+i}, \cdots, \xi^{n+i} \rangle$ for all $i \geq 0$.*

Proof. Let $E = \langle 1, \xi, \cdots, \xi^{n+i} \rangle$; and for $e \in E$, let $\bar{e}$ denote its image in E/mE. Then $\bar{\xi}^l + a_{l+1}\bar{\xi}^{l+1} + \cdots + a_n\bar{\xi}^n = 0$ implies $\bar{\xi}^l \in \langle \bar{\xi}^{l+1}, \cdots, \bar{\xi}^n \rangle$, $\bar{\xi}^{l+1} \in \langle \bar{\xi}^{l+2}, \cdots, \bar{\xi}^{n+1} \rangle, \cdots, \bar{\xi}^{l+i} \in \langle \bar{\xi}^{l+1+i}, \cdots, \bar{\xi}^{n+i} \rangle$. Therefore $\bar{E} = \langle 1, \bar{\xi}, \cdots, \bar{\xi}^{l-i}, \bar{\xi}^{l+1+i}, \cdots, \bar{\xi}^{n+i} \rangle$, and hence the lemma follows from Nakayama's lemma.   Q.E.D.

Before proceeding to the proof of Theorem 3.2, we shall mention a few well-known facts which will be used there. Let R, m be a quasi-local ring, and let M be a finitely generated R-module. It follows from Nakayama's lemma that any generating set for M contains a minimal generating set and that any two minimal generating sets contain the same number of elements ($= \dim M/mM$ as an R/m-module). Moreover, if M is free of rk n, then any minimal generating set is free [B(a), p. 109, Corollary].

Theorem 3.2 shows that for a quasi-local R, flatness of all the modules $\langle 1, \xi, \cdots, \xi^t \rangle$ is equivalent to the equivalent conditions of Lemma 2.1. These equivalent conditions imply that $R[X]/I$ ($= R[\xi]$) is R-flat and min I contains a regular element, as we have already noted in Lemma 2.1; but the converse is false, as Example 2.14(2) shows. If one imposes the additional condition that R be integrally closed, then by Corollary 2.13 the converse is valid. Recall also that a finitely generated module over a quasi-local ring is flat if and only if it is free [B(a), p. 167, Exercise 3-e], so that the word "flat" could equally well be substituted for "free" in Theorem 3.2.

3.2 **Theorem.** *Let R, m be a quasi-local ring, let I be a nonzero ideal of $R[X]$, and let ξ denote the equivalence class of X in $R[X]/I$. Then the following are equivalent*:

(i) *$\langle 1, \xi, \cdots, \xi^t \rangle$ is a free R-module for all $t \geq 0$.*

(ii) *$\langle 1, \xi, \cdots, \xi^t \rangle$ is a free R-module for some $t \geq 0$ for which $1, \xi, \cdots, \xi^t$ are linearly dependent over R.*

(iii) *$I = fR[X]$ for some $f \in R[X]$ whose leading coefficient is regular, and $c(I) = R$.* (Note: this is one of the equivalent assertions of Lemma 2.1.)

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): First consider the case that the residue field R/m is infinite. Since some ξ^j can be written as a linear combination of $1, \xi, \cdots, \hat{\xi}^j, \cdots, \xi^t$ there exists $f \in I$ such that $c(f) = R$. Choose f to have minimal degree among elements of I having content R, and let $n = \deg f$. *Claim. $f \in$ min I.* Since not all of the coefficients of f are in m and R/m is infinite, there exists $a \in R$ such that $f(a) = u \notin m$; and by replacing f by f/u we may assume $u = 1$. Moreover, by applying the R-automorphism of $R[X]$ given by $X \to X + a$, we may further assume that $a = 0$. Thus, we have reduced to the case that f has constant term 1. It follows that ξ is a unit of $R[\xi]$. By Lemma 3.1, $\langle 1, \xi, \cdots, \xi^{n+i} \rangle =$

$\langle \xi^{i+1}, \cdots, \xi^{n+i} \rangle$ for all $i \geq 0$. In particular, for $i = t - n$, $\langle 1, \xi, \cdots, \xi^{n+i} \rangle = \xi^{i+1} \langle 1, \xi, \cdots, \xi^{n-1} \rangle$ is free; and hence, since ξ is a unit of $R[\xi]$, it follows that $\langle 1, \xi, \cdots, \xi^{n-1} \rangle$ is free. Therefore, $1, \xi, \cdots, \xi^{n-1}$ must be linearly independent over R, for otherwise some proper subset forms a basis (because R is quasi-local), contradicting the choice of f. Thus $f \in \min I$. Therefore (iii) holds by Lemma 2.1.

It remains to remove the restriction that R/m be infinite. We do this by the device used in §2 of adjoining an indeterminate Y, thus replacing the ring R, m by $R(Y), mR(Y)$. There are then a few things that must be checked. For one, we have a commutative diagram

$$0 \to I \otimes_R R(Y) \to R[X] \otimes_R R(Y) \to (R[X]/I) \otimes_R R(Y) \to 0$$

$$0 \to IR(Y)[X] \quad \to \quad R(Y)[X] \quad \to \quad R(Y)[X]/IR(Y)[X] \to 0.$$

The top row is exact since $R(Y)$ is R-flat, and the vertical maps are $R(Y)$-algebra isomorphisms. Let E denote the R-module $\langle 1, \xi, \cdots, \xi^t \rangle$. Then E is R-free implies $E \otimes_R R(Y)$ is $R(Y)$-free; and since $R(Y)$ is R-flat, we have an injection $E \otimes_R R(Y) \to (R[X]/I) \otimes_R R(Y)$. Thus, (ii) is satisfied for the ring $R(Y)$ and the $R(Y)$-module $\langle 1 \otimes 1, \xi \otimes 1, \cdots, (\xi \otimes 1)^t \rangle$. Therefore by the above argument, there exists a regular element $f \in \min(IR(Y)[X])$ such that $IR(Y)[X] = fR(Y)[X]$, and hence, by Lemma 2.9, I is principal generated by a regular element of $\min I$. Also $c(I) = R$ by the same argument as above.

(iii) $\Rightarrow$ (i): Let $f = a_0 + a_1 X + \cdots + a_n X^n$. Since $c(f) = R$, we may assume $a_0, \cdots, a_{l-1} \in m$ and $a_l = 1$. Since $f \in \min I$, $\langle 1, \xi, \cdots, \xi^t \rangle$ is certainly free for each $t < n$; so assume $t = n + i$ for some $i \geq 0$. By Lemma 3.1, $\langle 1, \xi, \cdots, \xi^{n+i} \rangle = \langle 1, \xi, \cdots, \xi^{l-1}, \xi^{l+1+i}, \cdots, \xi^{n+i} \rangle$. *Claim.* $1, \xi, \cdots, \xi^{l-1}, \xi^{l+1+i}, \cdots, \xi^{n+i}$ are linearly independent over R. Suppose $b_0 + b_1 \xi + \cdots + b_{l-1} \xi^{l-1} + b_{l+1+i} \xi^{l+1+i} + \cdots + b_{n+i} \xi^{n+i} = 0$, $b_j \in R$, and let $g = b_0 + b_1 X + \cdots + b_{n+i} X^{n+i}$. Then $g \in I$; and since $I = fR[X]$, $g = hf$ for some $h \in R[X]$. Let $h = c_0 + c_1 X + \cdots + c_i X^i$ (the degree of h must be i since the leading coefficient of f is a regular element of R). Equating coefficients of $X^l, X^{l+1}, \cdots, X^{l+i}$ in $g = hf$, we get a set of $i + 1$ equations:

$$0 = c_0 a_l + c_1 a_{l-1} + \cdots + c_i a_{l-i}$$

$$0 = c_0 a_{l+1} + c_1 a_l + c_2 a_{l-1} + \cdots + c_i a_{l-i+1}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$0 = c_0 a_{l+i} + c_1 a_{l+i-1} + \cdots + c_i a_l.$$

(We define any a_i having subscript less than zero or greater than n in these

equations to be zero.) If we regard these equations as $i + 1$ equations in the $i + 1$ unknowns $c_0, \cdots, c_i$, the coefficient matrix has $a_1 = 1$ down the diagonal and elements of m above the diagonal, and hence its determinant is a unit of R. Therefore by Cramer's rule [L, p. 330], $c_0 = c_1 = \cdots = c_i = 0$. Thus $b = 0$, and hence $g = 0$. Q.E.D.

If one imposes the mild restriction that ξ be a regular element of $R[\xi]$ ($= R[X]/I$), then $1/\xi$ is in the total quotient ring of $R[\xi]$, and it thus makes sense to speak of the ring $R[1/\xi]$. Conditions (i) and (ii) of Theorem 3.2 can then be replaced by symmetric conditions involving $1/\xi$ since the finite R-modules $\langle 1, \xi, \cdots, \xi^t \rangle$ and $\langle (1/\xi)^t, (1/\xi)^{t-1}, \cdots, 1 \rangle$ are isomorphic. It follows that 3.2(i) implies $R[\xi]$ and $R[1/\xi]$ are both R-flat since these modules are then direct limits of flat modules. The question thus arises of whether conversely $R[\xi]$ and $R[1/\xi]$ are R-flat implies the equivalent conditions of Theorem 3.2. We shall next demonstrate that this is the case when min I contains a regular element.

The following lemma is well known in the case that R is a domain, since then a suitably chosen valuation ring has the required properties.

3.3 Lemma. *Let R be a ring, let A be a finitely generated ideal of R which contains a regular element, and let P be a prime ideal of R. Then there exists an R-algebra R' such that*

    (i) *R' is R-torsion-free,*

    (ii) *there exists a prime ideal P' of R' such that $P' \cap R = P$, and*

    (iii) *$AR' = aR'$ for some R'-regular element $a \in A$.*

Proof. By induction on the number of elements in a generating set for A which contains a regular element, it suffices to prove the lemma when $A = (a, b)$ with a regular. By localizing at P, we may assume R is quasi-local with maximal ideal P. Consider the ring $R' = R[b/a]$. R' satisfies (iii) since $(a, b)R' = aR'$. Since P is a maximal ideal of R, (ii) would be satisfied if we had $PR' \neq R'$. Therefore assume $PR' = R'$. Then $1 = x_1 f_1(b/a) + \cdots + x_t f_t(b/a)$, $x_i \in P$, $f_i \in R[X]$. From this equation, we conclude that $a^i \in P(a, b)^i$ for some $i \geq 1$. Therefore, $(a, b)^i = (a^{i-1}b, a^{i-2}b^2, \cdots, b^i) + P(a, b)^i$; so by Nakayama's lemma, $(a, b)^i = (a^{i-1}b, a^{i-2}b^2, \cdots, b^i)$, and hence $a^i \in (a^{i-1}b, a^{i-2}b^2, \cdots, b^i)$. Therefore there exist $r_j \in R$ such that $1 = r_1(b/a) + \cdots + r_i(b/a)^i$. It follows that b/a is a unit of R' and that a/b is integral over R. But then, by lying over, $PR[a/b] \neq R[a/b]$ and the ring $R[a/b]$ has the desired properties. Q.E.D.

We next need a rather technical lemma. Let $\phi: R[X] \to R[X]/I$ be the canonical homomorphism, and recall that $\xi = \phi(X)$ and that we are denoting the ring $\phi(R)[\xi]$ by merely $R[\xi]$. If ξ is regular in $R[\xi]$, then $\phi(R)[1/\xi]$ is a subring of $T(R[\xi])$; and we shall use I^* for the kernel of the homomorphism $R[X] \to \phi(R)[1/\xi]$ taking $X \to 1/\xi$. Then $a_0 X^n + a_1 X^{n-1} + \cdots + a_n$, $a_i \in R$, is in I^* if

and only if $a_0 + a_1X + \cdots + a_nX^n \in I$. Recall also that, for any nonempty nonzero subset H of $R[X]$, deg H (subd H) denotes the minimum of the degrees (subdegrees) of the nonzero elements of H.

3.4 Lemma. *Let I be an ideal of $R[X]$ such that min I contains a regular element, let P be a prime ideal of R, and let $\psi: R[X] \to (R/P)[X]$ be the homomorphism which reduces coefficients* mod P. *If ξ is regular in $R[\xi]$ and $R[\xi]$ is R-flat, then* deg I – deg $\psi(I) \leq$ subd $\psi(I^*)$.

Proof. Let f be a regular element of min I. By Lemma 3.3, there exists an R-algebra R' which is R-torsion-free, which has a prime P' lying over P, and such that $c_R(f)R' = aR'$ for some R'-regular element $a \in c_R(f)$. Also, $R'[X]/IR'[X] \cong (R[X]/I) \otimes_R R'$ is R'-flat by [**B**(a), p. 34, Corollary 2], and $f \cdot 1 \in \min(IR'[X])$ by Lemma 2.3 (where 1 without a subscript will denote the identity of $R'[X]$). Therefore by Lemma 2.1 there exists $f_1 \in R'[X]$ such that $IR'[X] = f_1R'[X]$ and the leading coefficient of f_1 is regular. Note also that, by Lemma 2.3, deg $I =$ deg $(IR'[X])$. If $f_1 = a_0 + a_1X + \cdots + a_nX^n$, $a_i \in R'$, $a_n \neq 0$, let $f_1^* = a_0X^n + a_1X^{n-1} + \cdots + a_n$. For any $g^* = b_0X^m + b_1X^{m-1} + \cdots + b_m \in I^*$, the corresponding polynomial $g = b_0 + b_1X + \cdots + b_mX^m \in I$. Therefore there exists $h \in R'[X]$ such that $g \cdot 1 = hf_1$. Then $g^* \cdot 1 = h^*f_1^*$. Since there exists a prime P' of R' lying over P, the homomorphism ψ "extends" to a homomorphism ψ' of $R'[X]$; and it follows that subd $\psi'(I^*R'[X]) \geq$ subd $\psi'(f_1^*) = n -$ deg $\psi'(f_1)$. Similarly, deg $\psi'(IR'[X]) \geq$ deg $\psi'(f_1)$ since $IR'[X] = f_1R'[X]$. Therefore, subd $\psi'(I^*R'[X]) \geq$ deg I – deg $\psi'(IR'[X])$. The desired conclusion follows since $\psi'(I^*R'[X]) = \psi(I^*)(R'/P')[X]$ and $\psi'(IR'[X]) = \psi(I)(R'/P')[X]$ provided we identify $(R/P)[X]$ with an appropriate subring of $(R'/P')[X]$.

3.5 Theorem. *Let R, m be a quasi-local ring, let I be a proper ideal of $R[X]$, and let ξ denote the equivalence class of X in $R[X]/I$. If ξ is a regular element of $R[\xi]$, then the following is equivalent to conditions* (i)–(iii) *of Theorem 3.2:*

(iv) min I *contains a regular element, and $R[\xi]$ and $R[1/\xi]$ are R-flat.*

Proof. ($\Longrightarrow$) (iii) of Theorem 3.2 implies min I contains a regular element, while (i) implies that $R[\xi]$ and $R[1/\xi]$ are R-flat, since these modules are direct limits of the modules in (i) [**B**(a), p. 28, Proposition 2].

($\Longleftarrow$) Let $f = a_0 + a_1X + \cdots + a_nX^n$, $a_n \neq 0$, be a regular element of min I, so a_n is a regular element of R. By Lemma 2.1, it suffices to show $c(f)$ is a principal ideal. Since R is quasi-local, we can choose a minimal set $a_{i_1}, \cdots,$ a_{i_t} of generators for $c(f)$ from among the a_i, and we do this in such a way that i_1 is as large as possible and $i_1 < i_2 < \cdots < i_t$. We shall assume $t > 1$ and show this leads to a contradiction.

There exist $b_{lj} \in R$ such that $a_l = \sum_{j=1}^{t} b_{lj} a_{i_j}$, $l = 0, 1, \cdots, n$; and we can choose $b_{i_1 1} = b_{i_2 2} = \cdots = b_{i_t t} = 1$. Moreover, since i_1 was chosen as large as possible, $b_{l1} \in m$ for $l > i_1$. Then

$$f = \left(\sum_{j=1}^{t} b_{0j} a_{i_j} \right) + \left(\sum_{j=1}^{t} b_{1j} a_{i_j} \right) X + \cdots + \left(\sum_{j=1}^{t} b_{nj} a_{i_j} \right) X^n$$

$$= a_{i_1}(b_{01} + b_{11}X + \cdots + b_{n1}X^n) + \cdots + a_{i_t}(b_{0t} + b_{1t}X + \cdots + b_{nt}X^n).$$

By applying the flatness of $R[\xi]$ via criterion 1.2(b), we conclude that there exist $c_i \in (a_{i_2}, \cdots, a_{i_t}) : a_{i_1} \subset m$ such that

$$(1) \qquad\qquad b_{01} + b_{11}\xi + \cdots + b_{n1}\xi^n = \sum_{i=0}^{p} c_i \xi^i.$$

Similarly, by applying the flatness of $R[1/\xi]$, there exist $d_i \in (a_{i_1}, \cdots, a_{i_{t-1}}) : a_{i_t} \subset m$ such that

$$(2) \qquad\qquad b_{0t}(1/\xi)^n + b_{1t}(1/\xi)^{n-1} + \cdots + b_{nt} = \sum_{i=0}^{q} d_i (1/\xi)^i.$$

If ψ denotes the homomorphism $R[X] \to (R/m)[X]$ which reduces coefficients mod m, we have from (1) that $\deg \psi(l) \le i_1$ since $b_{i_1 1} = 1$ and $b_{l1} \in m$ for $l > i_1$. Similarly, it follows from (2) and the fact that $b_{i_t t} = 1$ that subd $\psi(I^*) \le n - i_t$ (where I^* is the kernel of the homomorphism $R[X] \to R[1/\xi]$). Then $n - i_t \ge n - \deg \psi(l) \ge n - i_1$, the first inequality following from Lemma 3.4. Therefore $i_t \le i_1$, a contradiction to the assumption that $t > 1$.   Q.E.D.

We shall now globalize and summarize the preceding local results.

3.6 **Summary.** Let R be a ring, let I be an ideal of $R[X]$ such that min I contains a regular element, and let ξ denote the equivalence class of X in $R[X]/I$. Then the following are equivalent:

(i) $\langle 1, \xi, \cdots, \xi^t \rangle$ is a flat R-module for all $t \ge 0$.

(ii) $\langle 1, \xi, \cdots, \xi^t \rangle$ is a flat R-module for some $t \ge 0$ for which $1, \xi, \cdots, \xi^t$ are linearly dependent over R.

(iii) $c(I) = R$; and for every prime P of R, there exists $f \in R_P[X]$ such that the leading coefficient of f is regular in R_P and $IR_P[X] = fR_P[X]$.

(iv) $I = (f_1, \cdots, f_n)$, $f_i \in$ min I, and $c(I) = R$.

(v) $c(\text{min } I) = R$.

(vi) $R[\xi]$ is R-torsion-free and $c(g)$ is invertible for some $g \in$ min I (or for every regular $g \in$ min I).

If ξ is regular in $R[\xi]$, then (i)–(vi) are equivalent to

(vii) $R[\xi]$ and $R[1/\xi]$ are R-flat.

If R is integrally closed, then (i)–(vi) are also equivalent to

(viii) $R[\xi]$ is R-flat.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): Apply Theorem 3.2 and the fact that flatness localizes well [**B**(a), p. 116, Proposition 15].

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v): Apply Theorem 2.15.

(vi) $\Leftrightarrow$ (iii): Suppose (vi) is satisfied. Then for any prime P of R, $R[\xi]_P$ is R_P-torsion-free by Lemma 2.12. Recall that an ideal is invertible if and only if it is finitely generated, contains a regular element, and is locally principal [**B**(a), p. 148, Theorem 4]. Therefore $c(g)$ is invertible implies $c(g)_P$ is principal and is generated by a regular element. Moreover, $g/1 \in \min(I_P)$ by Lemma 2.3. Therefore (iii) follows from Lemma 2.1. Conversely, suppose (iii) is valid. For any regular $g \in \min I$, $g/1 \in \min(I_P)$ for any prime P of R, by Lemma 2.3. Then $c_{R_P}(g/1)$ is principal by Lemma 2.1, and hence $c(g)$ is locally principal and therefore invertible.

(vii) $\Leftrightarrow$ (iii): Apply Theorem 3.5.

(viii) $\Leftrightarrow$ (iii): Apply Corollary 2.13.   Q.E.D.

The simplest case to which the above results apply is the case where $\deg \min I = 1$, i.e. $\min I$ contains a regular element of degree 1, or equivalently, ξ is in the total quotient ring of R. Then $\xi = a/b$, $a, b \in R$; and $\langle 1, \xi, \cdots, \xi^t \rangle$ is isomorphic as an R-module to the ideal $(b^t, ab^{t-1}, \cdots, a^t) = (a, b)^t$. Moreover, $\langle 1, \xi, \cdots, \xi^t \rangle$ is flat if and only if the ideal $(a, b)^t$ is invertible [**B**(a), p. 148, Theorem 4]. Richman [**R**, p. 797, Proposition 3] has proved that if D is an integrally closed domain and $\xi = a/b$ is in the quotient field of D, then $D[\xi]$ is flat implies (a, b) is invertible, which may be regarded as a very special case of our (viii) $\Rightarrow$ (vi).

We have shown in Corollary 2.13 that if R is quasi-local integrally closed and $\min I$ contains a regular element, then $R[X]/I$ is R-flat implies I is principal. We conclude §3 with some generalizations of this result.

3.7 **Theorem.** *Let R, m be a quasi-local ring, and let I be an ideal of $R[X]$ such that $\min I$ contains a regular element. If $R[X]/I$ is R-flat, then $\overline{I}R[X]$ is principal. If in addition $J(\overline{R}) \subset R$, then I is principal.*

Proof. Let $f = a_0 + a_1 X + \cdots + a_n X^n$ be a regular element of $\min I$, and let t denote the largest integer such that $c(f) = (a_t, a_{t+1}, \cdots, a_n)$. We shall prove that $c(f)\overline{R} = a_t\overline{R}$. If $t = n$ this is immediate, so we may assume $t < n$. Then there exist $r_{ij} \in R$ such that

$$a_i = \sum_{j=t}^{n} r_{ij} a_j, \qquad i = 0, 1, \cdots, t - 1,$$

and

$$f = \sum_{j=t}^{n} a_j (r_{0j} + r_{1j} X + \cdots + r_{t-1\,j} X^{t-1} + X^j).$$

If ξ denotes the equivalence class of X in $\overline{R}[X]/I\overline{R}[X]$, then $f(\xi) = 0$ implies

$$r_{0t} + r_{1t} \xi + \cdots + r_{t-1\,t} \xi^{t-1} + \xi^t \in (a_{t+1}, \cdots, a_n) \overline{R}[\xi] : a_t$$

$$= ((a_{t+1}, \cdots, a_n) : a_t) \overline{R}[\xi] \subset m \overline{R}[\xi],$$

the equality being a consequence of the $\overline{R}$-flatness of $\overline{R}[\xi]$. (See Theorem 2.18 and Criterion 1.2(b).) Thus $g = r_{0t} + r_{1t} X + \cdots + r_{t-1\,t} X^{t-1} + X^t \in m\overline{R}[X] + I\overline{R}[X]$.

Let $\overline{m}$ be any maximal ideal of $\overline{R}$, and let $R' = \overline{R}_{\overline{m}}$, $m' = \overline{m} \overline{R}_{\overline{m}}$. Then $R'[X]/IR'[X]$ is R'-flat by [B(a), p. 116, Proposition 15], and $f \cdot 1 \in \min(IR'[X])$ by Lemma 2.3. Since R' is quasi-local and integrally closed in $T(\overline{R})_{\overline{m}}$ by [B(c), p. 22, Proposition 16] and since $R' \subset T(\overline{R})_{\overline{m}} \subset T(\overline{R}_{\overline{m}})$, $IR'[X]$ is generated by an element $b' \in \min(IR'[X])$ with $c_{R'}(b') = R'$ by Theorem 2.7. It follows from the content formula that $c(f)R'$ is principal and hence is generated by some regular element of the form $a_q \cdot 1$, where $t \le q \le n$ and 1 denotes the identity of R'. *Claim.* $q = t$. Let $\phi: R'[X] \to (R'/m')[X]$ be the homomorphism which reduces coefficients mod m'. Then $g \cdot 1 \in m'R'[X] + IR'[X]$ implies $\phi(g \cdot 1) \in \phi(b')(R'/m')[X]$. Since $\deg \phi(g \cdot 1) = t$, $\deg \phi(b') \le t$. But $\deg \phi(b') \ge q \ge t$, so therefore $q = t$. Thus, $c(f)R' = a_t R'$; and since this is true for an arbitrary maximal ideal $\overline{m}$ of $\overline{R}$, $c(f)\overline{R} = a_t \overline{R}$. Therefore by Lemma 2.1, $I\overline{R}[X] = \overline{f}\,\overline{R}[X]$, where $\overline{f} = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n$ and $a_i = \alpha_i a_t$, $\alpha_i \in \overline{R}$.

For the second assertion of the theorem, it will suffice to show that there exist $d_i \in R$ such that $\alpha_i \equiv d_i \bmod J$, $i = 0, \cdots, n$. That $\alpha_{t+1}, \cdots, \alpha_n$ are in J follows from our previous observation that, for any maximal ideal $\overline{m}$ of $\overline{R}$, if $c(f)\overline{R}_{\overline{m}} = a_q \overline{R}_{\overline{m}}$ for $t \le q \le n$, then $q = t$. It remains only to show that for $i \le t$ we have $\alpha_i \equiv d_i \bmod J$ for some $d_i \in R$. Let $\psi: \overline{R}[X] \to (\overline{R}/J)[X]$ be the map which reduces coefficients mod J. Then since $r_{0t} + r_{1t} X + \cdots + r_{t-1\,t} X^{t-1} + X^t \in I\overline{R}[X] + J\overline{R}[X]$ and $I\overline{R}[X] = \overline{f}\,\overline{R}[X]$, we have $\psi(r_{0t} + r_{1t} X + \cdots + X^t) \in \psi(\overline{f})(\overline{R}/J)[X]$. But $\psi(r_{0t} + r_{1t} X + \cdots + X^t)$ and $\psi(\overline{f})$ have the same degree and are both monic, so $\psi(r_{it}) = \psi(\alpha_i)$ for $i = 0, \cdots, t$.   Q.E.D.

Note that the hypothesis that $R[X]/I$ is R-flat in the above theorem is equivalent to the (apparently weaker) hypothesis that $\overline{R}[X]/I\overline{R}[X]$ is $\overline{R}$-flat, by Theorem 2.18. Also, the hypothesis $J(\overline{R}) \subset R$ is not as contrived as it might at first seem; for at least in the rather trivial case of a 1-dimensional quasi-local domain D, m, it can be seen that $J(\overline{D}) \subset D$ if (and only if), for every ideal I of $D[X]$, $D[X]/I$ is D-flat implies I is principal.

The proof of this remark (which is inspired by [E₃, p. 345]) requires a few

preliminary observations. For elements a, b in a ring R we denote by $(bR : a)$ the ideal $\{r \in R \mid ra \in bR\}$. A criterion of [A, p. 801, Theorem 1] says that an overring R' of R, $R' \subset T(R)$, is R-flat iff $(bR : a)R' = R'$ for every $a/b \in R'$; and in the case of a simple extension $R' = R[x/y]$, it is easily seen that this criterion reduces to the following: $R[x/y]$ is R-flat if and only if $(yR : x)R[x/y] = R[x/y]$, or equivalently, if and only if $\sqrt{(yR : x)}R[x/y] = R[x/y]$. Also, if x/y is a unit of $T(R)$, if I is the kernel of the R-homomorphism $R[X] \to R[x/y]$ defined by $X \to x/y$, and if $y/x \in \overline{R} \setminus R$, then x/y, $y/x \notin R$. Therefore, $(x) \neq (x, y) \neq (y)$; and hence if R is quasi-local, then (x, y) is not principal and it follows from Lemma 2.1 that I is not principal.

Now let D, m be a 1-dimensional quasi-local domain such that $J(\overline{D}) \not\subset D$. To prove the above remark we need only find $x/y \in T(D)$ satisfying an equation of the form $1 + r_1(x/y) + \cdots + r_n(x/y)^n = 0$, with $r_i \in m$ ($= \sqrt{(yR : x)}$), and such that $y/x \notin D$; for then $y/x \in \overline{D} \setminus D$ and $\sqrt{(yR : x)}D[x/y] = D[x/y]$. If $m\overline{D} > m$ then $m\overline{D} \not\subset D$ and there exist $a/b \in \overline{D}$ and $c \in m$ such that $ca/b \notin D$. Then $(ca/b)^n + cr_1(ca/b)^{n-1} + \cdots + c^n r_n = 0$ for some $r_i \in D$; and hence $x/y = b/ac$ has the required properties. Similarly, if $m\overline{D} = m$, then there exists $a/b \in J(\overline{D}) \setminus D$ such that $(a/b)^n \in m$ for some n, and in this case $x/y = b/a$ has the required properties.

The next proposition is another generalization of Corollary 2.13. A similar statement for noetherian rings can be found in [Na$_2$, p. 445, Corollary 4.3(iv)].

3.8 Proposition. *Let R be quasi-local, let A be an ideal of R contained in the nil radical of R, and let I be an ideal of $R[X]$ such that* min I *contains a regular element. If R/A is integrally closed and $R[X]/I$ is R-flat, then I is principal.*

Proof. Let $M = R[X]/I$. Since M is R-flat, $0 \to I/AI \to (R/A)[X] \to M/AM \to 0$ is exact and M/AM is R/A-flat. By Corollary 2.13 and Lemma 2.3, I/AI is a principal ideal of $(R/A)[X]$. Since I is a finitely generated ideal by Theorem 2.19, it follows from Nakayama's lemma that I is principal.

4. The case of arbitrary I. Most of the theorems of §2 and §3 involve the hypothesis that min I contains a regular element, and indeed we have shown by examples that the theorems are false without this hypothesis. One can ask, however, to what extent this condition can be weakened, and in particular, for what rings R the condition can be dropped. In studying these questions, we avoid the difficulties of §2 and §3 by means of various finiteness assumptions on either I or $R[X]/I$. Thus, although the results of this section are somewhat superficial, they nonetheless give an interesting perspective to the general problem.

We remind the reader that, for any R-module M, free $\Rightarrow$ projective $\Rightarrow$ flat. Moreover, free $\Leftrightarrow$ projective for R quasi-local; and if M is finitely generated,

then projective $\Leftrightarrow$ flat for a large class of rings including domains, quasi-semi-local rings, and noetherian rings. We refer the reader to [B(a)] for these facts.

As usual, $R[X]$ denotes the polynomial ring in one indeterminate over R. We shall begin with a few simple remarks on idempotents.

4.1 Lemma. *Let A be an ideal of R, and consider the following statements:*

(i) *A is generated by an idempotent.*

(ii) *A is locally trivial.*

(iii) *$A = A^2$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii); *and if A is finitely generated, then* (iii) $\Rightarrow$ (i). *(In general, (iii) $\not\Rightarrow$ (ii). Take A to be the maximal ideal of a nondiscrete rank 1 valuation ring.)*

Proof. (i) $\Rightarrow$ (ii) follows from the observation that, for any idempotent e of a quasi-local ring, either $e = 0$ or $e - 1 = 0$; (ii) $\Rightarrow$ (iii) since two ideals with the same localizations are equal; and (iii) $\Rightarrow$ (i) when A is finitely generated by [B(a), p. 83, Corollary 3]. (See also [B(a), p. 172, Exercise 15].)

4.2 Lemma. *Let R' be a polynomial ring over R, and let I be an ideal of R'. If $c(I)$ satisfies 4.1(ii), then $\mathrm{Supp}\, I$ is open; while if $c(I)$ satisfies 4.1(i), then $\mathrm{Supp}\, I$ is open and closed. (Recall that $\mathrm{Supp}\, I$ denotes the set of primes P' of R' such that $I_{P'} \neq 0$ [B(a), p. 132].)*

Proof. Let P' be a prime of R', and let $P = P' \cap R$. If $c(I)$ satisfies (ii), $I_{P'} \neq 0 \Leftrightarrow I_P \neq 0 \Leftrightarrow c(I_P) = R_P \Leftrightarrow c(I) \not\subset P \Leftrightarrow c(I)R' \not\subset P'$, the nontrivial implications all following from the fact that $c(I_P) = R_P$ implies I_P contains a regular element. If in addition $c(I) = eR$ for an idempotent e, then $c(I) \not\subset P \Leftrightarrow 1 - e \in P \Leftrightarrow (1 - e)R' \subset P'$.   Q.E.D.

If I is a locally free ideal of R', then rk $I = 0$ or 1. For such an ideal, Lemma 4.2 asserts that if $c(I)$ is generated by an idempotent, then rk $I = 1$ on an open and closed set and hence rk I is locally constant. We need this observation in the proof of the next theorem.

4.3 Theorem. *Let I be an ideal of $R[X]$. Then the following are equivalent:*

(i) *I is a projective ideal of $R[X]$, and $c(I)$ is generated by an idempotent.*

(ii) *$R[X]/I$ is R-flat, and I is a finitely generated ideal of $R[X]$.*

(iii) *I is a finitely generated flat ideal of $R[X]$, and $c(I)$ is generated by an idempotent.*

Proof. (i) $\Rightarrow$ (ii): Since I is $R[X]$-projective, it is locally free at primes of $R[X]$ by [Ka$_2$, p. 374, Theorem 2]. Therefore by Theorem 1.5, $R[X]/I$ is R-flat. By Lemma 4.2, rk I is locally constant, and hence I is finitely generated by [V$_2$, p. 431, Proposition 1.4].

(ii) $\Rightarrow$ (i): By Proposition 1.6, I is locally free; and by Corollary 1.3 and Lemma 4.1, $c(I)$ is generated by an idempotent. By the remark following Lemma 4.2, rk I is locally constant, and hence by [**B**(a), p. 138, Theorem 1], I is projective.

(i) and (ii) $\Rightarrow$ (iii): Projective implies flat by [**B**(a), p. 28].

(iii) $\Rightarrow$ (ii): If I is $R[X]$-flat, then it is locally free [**B**(a), p. 116, Proposition 15; and p. 167, Exercise 3-e]. Therefore, $R[X]/I$ is R-flat by Theorem 1.5.   Q.E.D.

Note that the above proof would carry through for a polynomial ring in arbitrarily many indeterminates except for the appeal to Proposition 1.6 in (ii) $\Rightarrow$ (i). Also, Example 2.21 shows that the various finiteness hypotheses of Theorem 4.3 are essential.

4.4 Theorem. *Let* R' *be a finitely generated* R-*algebra. If* I *is an ideal of* R' *such that* R'/I *is a finite projective* R-*module, then* I *is a finitely generated ideal.*

Proof. Let $R' = R[x_1, \cdots, x_n]$. Since R'/I is a finite R-module, the image of x_i in R'/I satisfies an R-integral equation, and hence there exists a monic polynomial $f_i(X) \in R[X]$ such that $f_i(x_i) \in I$. If L denotes the ideal of R' generated by $f_1(x_1), \cdots, f_n(x_n)$, then R'/L is a finite R-module. The sequence $0 \to I/L \to R'/L \to R'/I \to 0$ is an exact sequence of R-modules; and since R'/I is R-projective, it splits. Therefore, R'/L is finitely generated implies I/L is also a finitely generated R-module. Pre-images in I of the generators of I/L together with $f_1(x_1), \cdots, f_n(x_n)$ then form a generating set for I.   Q.E.D.

Vasconcelos [**V**$_2$, p. 432, Theorem 2.1] has shown that the following are equivalent for a ring R:

(4.5)

    (1) Finitely generated flat R-modules are projective.

    (2) Locally finitely generated projective R-modules are finitely generated.

    (3) Projective ideals of R are finitely generated.

Rings satisfying (4.5) include noetherian rings, quasi-semi-local rings, and domains; and, in keeping with the terminology of [**CP**], we shall call these rings $A(0)$ *rings*. We can obtain as a corollary to Theorem 4.4 a rather interesting characterization of such rings, but first we need a well-known intermediate characterization.

4.6 Lemma. *The following property of a ring* R *is equivalent to the properties of (4.5):*

    (4) *Every locally trivial ideal of* R *is finitely generated.*

Proof. (1) $\Rightarrow$ (4): Let A be locally (0) or R_P. Then R/A is locally free

and hence R-flat. Therefore by (1), R/A is R-projective. Thus the sequence $0 \to A \to R \to R/A \to 0$ splits, and it follows that A is principal.

(4) $\Rightarrow$ (1): Let M be a finite flat R-module. The invariant factors of M are locally (0) or R_P by [V_1, p. 506] and hence are finitely generated by (4). Therefore, M is projective by [V_1, p. 506, Proposition 1.3].

4.7 **Theorem.** *The following property of a ring R is equivalent to the properties of* (4.5):

(5)   *If I is an ideal of $R[X]$ such that $R[X]/I$ is a finite flat R-module, then I is a finitely generated ideal.*

Proof. By Theorem 4.4, (1) $\Rightarrow$ (5). It remains to show (5) $\Rightarrow$ (4). Let A be as in (4), and let $I = AR[X] + XR[X]$. I is finitely generated if and only if A is, so we must show I is finitely generated. $R[X]/I \cong R/A$; and since A is locally (0) or (1), R/A is locally free and hence is R-flat. Therefore by (5), I is finitely generated. Q.E.D.

Note that the proof of Theorem 4.7 shows that "finite flat" can be replaced by "cyclic flat" in the statement of (5) and that $R[X]$ can be replaced by a polynomial ring in finitely many indeterminates. Of course, the real problem is to replace "finite flat" by merely "flat" in 4.7. Thus,

Question. Does every $A(0)$ ring R have the property that if I is an ideal of $R[X]$ such that $R[X]/I$ is R-flat then I is a finitely generated ideal? The main theorem of §2 shows that the answer is "yes" when R is a domain. The next case to investigate should probably be that of a quasi-local ring R.

The problem can be further weakened to ask whether, for a ring R, $R[X]/I$ is R-flat implies I is locally finitely generated at primes of $R[X]$.[3] An affirmative answer to this question would allow us to delete part of the hypothesis of Proposition 1.6 and thus make it a true converse to Theorem 1.5. Moreover, Proposition 1.6 shows that an affirmative answer would also imply that I is then locally free at primes of $R[X]$. Under the stronger hypothesis that $R[X]/I$ is R-projective and $c(I) = R$, the corollary to the next proposition shows that I is, in fact, finitely generated.

4.8 **Proposition.** *Let R, m be a quasi-local ring, and let I be a nonzero ideal of $R[X]$. Then $R[X]/I$ is R-free (if and) only if there exists a monic $f \in R[X]$ such that $I = fR[X]$.*

Proof. $c(I) = 0$ or R by Corollary 1.3; and since $I \neq 0$, $c(I) = R$. Let $M = R[X]/I$. Since M is free and R is quasi-local, $\text{rk}_R M = \text{rk}_{R/m} M/mM$ [L, p. 418,

_____

(3) Heinzer and Ohm have now proved this—see footnote 2. They also give an example involving a quasi-local R which shows that the above question must be modified.

Proposition 8]. Moreover, since M is R-flat, the sequence $0 \rightarrow I/mI \rightarrow (R/m)[X]$ $\xrightarrow{\psi} M/mM \rightarrow 0$ of R/m-modules is exact. But $c(I) = R$ implies $I/mI \neq 0$, and hence ψ is not an isomorphism. It follows that M/mM is a finite R/m-module. Then by Theorem 3.2, I is principal and generated by some $f \in R[X]$ whose lead-ing coefficient is regular. Also, M finitely generated implies I contains a monic polynomial and hence is generated by a monic polynomial.

4.9 Corollary. *Let R be a ring, and let I be an ideal of $R[X]$. If $c(I) = R$ and $R[X]/I$ is R-projective, then $R[X]/I$ is R-finite and I is a finitely generated ideal.*(4)

Proof. $M = R[X]/I$ is R-projective implies M_P is R_P-free for every prime P of R. Since $c(I_P) = R_P$, $I_P \neq 0$ and hence, by 4.8, M_P has a free basis of the form $1, \xi/1, \cdots, (\xi/1)^t$, where $R[X]/I = R[\xi]$. But the hypothesis $c(I) = R$ im-plies rk M is bounded, so there exists t such that $1, \xi, \cdots, \xi^t$ generate M local-ly at P for every prime P. Therefore, $M = \langle 1, \xi, \cdots, \xi^t \rangle$, and then I is finitely generated by Theorem 4.4.   Q.E.D.

The content of the paper $[V_3]$ of Vasconcelos is the following theorem:

Theorem. *Let R be a noetherian ring. Then I is a projective ideal of $R[X]$ with $c(I)$ generated by an idempotent if and only if $R[X]/I$ is R-flat. Moreover, if $c(I) = R$ and $R[X]/I$ is R-projective, then $R[X]/I$ is R-finite.*

The appropriate nonnoetherian generalization of the first assertion is (i) $\Leftrightarrow$ (ii) of our Theorem 4.3, while Corollary 4.9 generalizes the second assertion.

5. Nagata's theorem. Let R be a ring, let $X_1, \cdots, X_n$ be indeterminates, and let I be an ideal of $R[X_1, \cdots, X_n]$ such that $R[X_1, \cdots, X_n]/I$ is R-flat. Nagata [Na$_1$, p. 164, Theorem 3] has shown that if R is a valuation ring, then I is a finitely generated ideal; and it was this theorem that first aroused our interest in the questions treated in this paper. Nagata's proof of this theorem actually gives considerably more information than the theorem states, and we shall devote this section to an analysis and elaboration of his proof.

We begin with a generalization of Nakayama's lemma. Recall that $J(R)$ denotes the J-radical of R, i.e., the intersection of the maximal ideals of R.

5.1 Lemma. *Let M be an R-module, let A be an ideal of R contained in $J(R)$, and let M' be a submodule of M such that $M = M' + AM$. If $M = \bigoplus M_i$, where $\{M_i\}$ is a family of finitely generated submodules of M, and M' is homo-geneous, then $M = M'$.*

---

(4) We are indebted to W. Heinzer for showing us how to delete an extraneous hypoth-esis from the original version of this corollary.

Proof. Let $M_i' = M' \cap M_i$. Since $M = M' + AM$ and M' is homogeneous, it follows that $M_i = M_i' + AM_i$. But then $M_i = M_i'$ by the usual Nakayama lemma, and hence $M = M'$.   Q.E.D.

If N is a finitely generated flat R-module such that N/JN is a projective (R/J)-module (where $J = J(R)$), then N is R-projective $[\mathbf{V}_1$, p. 508, Theorem 2.1]. Thus, if R/J satisfies the properties of (4.5), then R does also; and in particular, R satisfies (4.5) if R/J is noetherian. This observation will be used in the proof of the next theorem. This theorem includes the statement that if R/J is noetherian and I is any *homogeneous* ideal of $R[X_1, \cdots, X_n]$ such that $R[X_1, \cdots, X_n]/I$ is R-flat, then I is finitely generated.

5.2 Theorem. *Let* $F = \bigoplus F_i$, $i = 0, 1, 2, \cdots$, *be a graded ring which is a finitely generated algebra over* F_0. *If* $F_0/J(F_0)$ *is noetherian and* M *is a homogeneous ideal of* F *such that* F/M *is a flat* F_0-*module, then* M *is a finitely generated ideal of* F.

Proof. Let $J = J(F_0)$. Then F/JF is a finitely generated algebra over F_0/J, and therefore F/JF is noetherian and $M/(JF \cap M)$ is a finitely generated ideal of F/JF. Taking pre-images and using the fact that M is homogeneous, we conclude that M contains a finitely generated homogeneous ideal M' of F such that $M = M' + (JF \cap M)$. Since F/M is F_0-flat, $JF \cap M = JM$; and hence $M = M' + JM$. Let $M_i = F_i \cap M$. Since M is homogeneous, $F/M \cong \bigoplus_{i \geq 0} (F_i/M_i)$. Thus, each F_i/M_i is F_0-flat [B(a), p. 28, Proposition 2]. Also, each F_i is a finite F_0-module by [B(b), p. 10, Corollary]. Thus, as remarked above, F_i/M_i is F_0-projective for each $i \geq 0$, and so each of the sequences $0 \to M_i \to F_i \to F_i/M_i \to 0$ splits. Hence each M_i is a finite F_0-module; so $M = M'$ by Lemma 5.1.   Q.E.D.

Example 2.21 shows that the assumption that $F_0/J(F_0)$ is noetherian, or some such condition, is needed in Theorem 5.2. However, as we have mentioned in §4, it seems quite conceivable to us, at least in the case of a polynomial ring in one indeterminate, that the homogeneous assumption can be deleted from 5.2. The next theorem shows that this can be done provided one imposes a somewhat stronger hypothesis than F/M being F_0-flat. The proof of this involves the customary device of homogenizing by introducing a new indeterminate, and then applying Theorem 5.2.

5.3 Theorem. *Let* $F = \bigoplus F_i$, $i = 0, 1, 2, \cdots$, *be a graded ring which is a finitely generated algebra over* F_0, *let* M *be an ideal of* F, *and let* $\phi: F \to F/M$ *be the canonical homomorphism. If* $F_0/J(F_0)$ *is noetherian and* $\Sigma_{i=0}^{k} \phi(F_i)$ *is a flat* F_0-*module for each* $k \geq 0$, *then* M *is a finitely generated ideal.*

Proof. Let Z be an indeterminate; let $F^b = \bigoplus F_k^b Z^k$, where $F_k^b = \Sigma_{i=0}^{k} F_i$; and let $E^b = \bigoplus E_k^b Z^k$, where $E_k^b = \Sigma_{i=0}^{k} \phi(F_i)$. If $\phi^b: F^b \to E^b$ denotes the

graded F_0-algebra homomorphism defined by $\phi^b(\Sigma\, a_i Z^i) = \Sigma\, \phi(a_i) Z^i$, then the kernel M^b of ϕ^b is a homogeneous ideal of F^b; and it follows from Theorem 5.2 that M^b is finitely generated. Since M is the homomorphic image of M^b under the homomorphism which takes Z to 1, it follows that M is finitely generated.   Q.E.D.

We shall conclude by discussing Theorem 5.3 in some special cases. Consider first the case of a polynomial ring $R[X]$ in one indeterminate. If I is an ideal of $R[X]$ and ξ denotes the equivalence class of X in $R[X]/I$, then the modules $\Sigma_{i=0}^k \phi(F_i)$ of Theorem 5.3 are just the modules $\langle 1, \xi, \cdots, \xi^k \rangle$ studied in §3. We have proved there that if R is integrally closed and min I contains a regular element, then $R[\xi]$ is R-flat implies the modules $\langle 1, \xi, \cdots, \xi^k \rangle$, $k \geq 0$, are also (and we have shown that the integral closure and the condition on min I are indispensable). W. Heinzer has pointed out to us that the corresponding statement for a polynomial ring in 2 indeterminates over an integrally closed ring is no longer valid. However, in the notation of 5.3, when F_0 is a valuation ring, then F/M is flat implies the modules $\Sigma_{i=0}^k \phi(F_i)$ are flat, since over a valuation ring any submodule of a flat module is flat; and this is why Nagata's theorem follows from Theorem 5.3. More generally, if R is any ring such that (a) $R/J(R)$ is noetherian and (b) submodules of flat modules are flat, then it follows from Theorem 5.3 that $R[X_1, \cdots, X_n]/I$ is R-flat implies I is a finitely generated ideal of $R[X_1, \cdots, X_n]$. Any finite direct sum of Prüfer domains satisfies (b); and if R is a ring satisfying (a) and (b), then R is a finite direct sum of Prüfer domains (see $[E_2,$ pp. 116–117, Theorem 5 and Corollary]). Note also that there exist Prüfer domains which are neither noetherian nor quasi-semi-local and which satisfy (a) (see $[EO,$ p. 348, Example 4.5]).

BIBLIOGRAPHY

[A] T. Akiba, *Remarks on generalized rings of quotients*, Proc. Japan Acad. **40** (1964), 801–806.   MR **31** #4807.

[B] N. Bourbaki, *Éléments de mathématique.* XXVII et XXX. *Algèbre commutative.* Chaps. 1–6, Actualités Sci. Indust., nos. 1290, 1293, 1308, Hermann, Paris, 1961, 1964. MR **30** #2027; MR **33** #2660; MR **36** #146 [(a) Chaps. 1 and 2, (b) Chaps. 3 and 4, (c) Chaps. 5 and 6].

[C] L. Claborn, *Every abelian group is a class group*, Pacific J. Math. **18** (1966), 219–222.   MR **33** #4085.

[CP] S. H. Cox, Jr. and R. L. Pendleton, *Rings for which certain flat modules are projective*, Trans. Amer. Math. Soc. **150** (1970), 139–156.   MR **41** #6906.

[E₁] S. Endo, *On flat modules over commutative rings*, J. Math. Soc. Japan **14** (1962), 284–291.   MR **31** #3475.

[E₂] ———, *On semi-hereditary rings*, J. Math. Soc. Japan **13** (1961), 109–119. MR **25** #2107.

[E₃] ———, *Projective modules over polynomial rings*, J. Math. Soc. Japan **15** (1963), 339–352.   MR **27** # 5808.

[EO] D. Estes and J. Ohm, *Stable range in commutative rings*, J. Algebra **7** (1967), 343–362.   MR **36** #147.

[G] R. W. Gilmer, *Multiplicative ideal theory*, Queen's Papers in Pure and Appl. Math., no. 12, Queen's University, Kingston, Ont., 1968.   MR **37** #5198.

[Ka₁] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR 40 #7234.

[Ka₂] ———, *Projective modules*, Ann. of Math. (2) **68** (1958), 372–377.   MR 20 #6453.

[L] S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.   MR 33 #5416.

[Kr] W. Krull, *Beitrage zur Arithmetik kommutative Integritatsbereiche*, Math. Z. 41 (1936), 545–577.

[Na₁] M. Nagata, *Finitely generated rings over a valuation ring*, J. Math. Kyoto Univ. 5 (1966), 163–169.   MR 33 #1309.

[Na₂] ———, *Flatness of an extension of a commutative ring*, J. Math. Kyoto Univ. 9 (1969), 439–448.   MR 41 #191.

[Na₃] ———, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962.   MR 27 #5790.

[No] D. G. Northcott, *A generalization of a theorem on the content of polynomials*, Proc. Cambridge Philos. Soc. 55 (1959), 282–288.   MR 22 #1600.

[OR] J. Ohm and D. Rush, *Content modules and algebras*, Math. Scand. 30 (1972).

[R] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. 16 (1965), 794–799.   MR 31 #5880.

[S] R. Swan, *The number of generators of a module*, Math. Z. 102 (1967), 318–322. MR 36 #1434.

[V₁] W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. 138 (1969), 505–512.   MR 42 #252.

[V₂] ———, *On projective modules of finite rank*, Proc. Amer. Math. Soc. 22 (1969), 430–433.   MR 39 #4134.

[V₃] ———, *Simple flat extensions*, J. Algebra 16 (1970), 105–107.   MR 42 #252.

[ZS] O. Zariski and P. Samuel, *Commutative algebra*. Vols. 1, 2, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1958, 1960.   MR 19, 833;   MR 22 #11006 [(a) Vol. 1 (1958), (b) Vol. 2 (1960)].

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803

Current address (David E. Rush): Department of Mathematics, University of California, Riverside, California 92502